International Journal of Algebra, Vol. 7, 2013, no. 5, 245 - 250
HIKARI Ltd, www.m-hikari.com

Projective Resolutions and the Homology of an Induced Group

O. S. Olusa

Department of Mathematics
Adekunle Ajasin University
Akungba, Nigeria
olusafem@hotmail.co.uk

S. A. Ilori

Department of Mathematics
University of Ibadan
Ibadan, Nigeria
ailori@mailandnews.com

M. EniOluwafe

Department of Mathematics
University of Ibadan
Ibadan, Nigeria
michael.enioluwafe@gmail.com

Abstract

It is proved that the homology of an induced abelian group with coefficients in the different $G-$ modules occuring in its projective resolution are isomorphic.

Mathematics Subject Classification: 18G35, 20J05

Keywords: Projective resolution, Induced group, Homology group

1 Introduction

The projective resolutions of R – modules are of vital importance in the study of (co)homology of groups as evident in literature such as [2],[3].

This work is an investigation of the homology of an induced abelian group with coefficients in the subgroups occuring in its free resolution. It is known that an abelian group G admits a resolution of length at most 1. [2]

The objective here is to investigate some properties of the homology of the group G with coefficients in K and H appearing in the said resolution.

The methods of homological algebra presented in [2] are used in this study. To do this, we consider the abelian group G as a $\mathbb{Z}$– module and the groups K and H as G– modules.

We see that the application of the torsion functor on the projective resolutions ultimately yields some vital exact sequences.

Using these exact sequences it is proved here that the homology groups of the induced abelian group G with coefficients in the different G– modules K and H are isomorphic.

2 Projective Resolutions and Homology of Groups

Definition 2.1 *Let $P = \{P_n, \partial_n\}$ be a positive exact chain complex of projective (free) R– modules i.e. $\ni H_n(P) = 0 \ \forall \quad n \geq 1$ and let us assume that it also satisfies that $H_n(P) \cong M$. We will write it as follows*

$$P: \quad \cdots \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

and call it a projective (free) resolution of an R– module M.

Definition 2.2 *The n-th homology group of a group G with coefficients in a left $G-$module N is*

$$H_n(G; N) = Tor_n^{\mathbb{Z}G}(\mathbb{Z}, N)$$

3 Homology of an Induced Group

The homology group as defined above depends of course on the G-modules from which the coefficients are drawn. Thus for example we have $H_1(G, N) \cong N \otimes_{\mathbb{Z}} G/[G, G]$ for any trivial G-module, and in the particular case where $N = \mathbb{Z}$, $H_1(G, \mathbb{Z}) \cong G/[G, G]$ (see page 20 of [2]). The result that follows has to do with the isomorphism of homology group of a group with coefficients in different G-modules.

Theorem 3.1 *Let G be an induced abelian group of the form $G = \mathbb{Z}G \otimes_{\mathbb{Z}} B$ (B a subgroup of G) admitting a resolution*

$$0 \longrightarrow K \longrightarrow H \longrightarrow G \longrightarrow 0 \qquad (1)$$

Then

$$H_n(G, K) \cong H_n(G, H) \quad \forall \quad n \geq 1$$

Proof: Let G be an abelian group admitting the resolution (1). Then by Proposition 2.4 of [2] the abelian groups K, H and G possess projective resolutions $P(K), P(H), P(G)$ respectively as follows:

$$P(K) : \cdots \longrightarrow P_{kn} \xrightarrow{\partial_n} P_{k(n-1)} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} P_{k1} \xrightarrow{\partial_1} P_{k0} \xrightarrow{\epsilon} K \longrightarrow 0$$
$$P(H) : \cdots \longrightarrow P_{hn} \xrightarrow{\partial'_n} P_{h(n-1)} \xrightarrow{\partial'_{n-1}} \cdots \xrightarrow{\partial'_2} P_{h1} \xrightarrow{\partial'_1} P_{h0} \xrightarrow{\epsilon'} H \longrightarrow 0$$
$$P(G) : \cdots \longrightarrow P_{gn} \xrightarrow{\partial''_n} P_{g(n-1)} \xrightarrow{\partial''_{n-1}} \cdots \xrightarrow{\partial''_2} P_{g1} \xrightarrow{\partial''_1} P_{g0} \xrightarrow{\epsilon''} G \longrightarrow 0$$

We then obtain a short exact sequence of chain complexes which can be displayed as follows:

$$\begin{array}{ccccccccc}
0 & \rightarrow & P(K) & \stackrel{\varphi}{\rightarrow} & P(H) & \stackrel{\varphi'}{\rightarrow} & P(G) & \rightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & P_{kn} & \stackrel{\varphi_n}{\rightarrow} & P_{hn} & \stackrel{\varphi'_n}{\rightarrow} & P_{gn} & \rightarrow & 0 \\
 & & \downarrow \partial_n & & \downarrow \partial'_n & & \downarrow \partial''_n & & \\
0 & \rightarrow & P_{k(n-1)} & \stackrel{\varphi_{n-1}}{\rightarrow} & P_{h(n-1)} & \stackrel{\varphi'_{n-1}}{\rightarrow} & P_{g(n-1)} & \rightarrow & 0 \\
 & & \downarrow \partial_{n-1} & & \downarrow \partial'_{n-1} & & \downarrow \partial''_{n-1} & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow \partial_2 & & \downarrow \partial'_2 & & \downarrow \partial''_2 & & \\
0 & \rightarrow & P_{k1} & \stackrel{\varphi_1}{\rightarrow} & P_{h1} & \stackrel{\varphi'_1}{\rightarrow} & P_{g1} & \rightarrow & 0 \\
 & & \downarrow \partial_1 & & \downarrow \partial'_1 & & \downarrow \partial''_1 & & \\
0 & \rightarrow & P_{k0} & \stackrel{\varphi_0}{\rightarrow} & P_{h0} & \stackrel{\varphi'_0}{\rightarrow} & P_{g0} & \rightarrow & 0 \\
 & & \downarrow \epsilon & & \downarrow \epsilon' & & \downarrow \epsilon'' & & \\
0 & \rightarrow & K & \stackrel{\alpha}{\rightarrow} & H & \stackrel{\beta}{\rightarrow} & G & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \qquad (2)$$

Applying the functor $\mathbb{Z} \otimes_{\mathbb{Z}G} -$ to (2) gives:

$$
\begin{array}{cccccccc}
0 & \to & \mathbb{Z}\otimes_{\mathbb{Z}G} P(K) & \to & \mathbb{Z}\otimes_{\mathbb{Z}G} P(H) & \to & \mathbb{Z}\otimes_{\mathbb{Z}G} P(G) & \to & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{Z}\otimes_{\mathbb{Z}G} P_{kn} & \to & \mathbb{Z}\otimes_{\mathbb{Z}G} P_{hn} & \to & \mathbb{Z}\otimes_{\mathbb{Z}G} P_{gn} & \to & 0 \\
 & & \downarrow 1\otimes\partial_n & & \downarrow 1\otimes\partial'_n & & \downarrow 1\otimes\partial''_n & & \\
0 & \to & \mathbb{Z}\otimes_{\mathbb{Z}G} P_{k(n-1)} & \to & \mathbb{Z}\otimes_{\mathbb{Z}G} P_{h(n-1)} & \to & \mathbb{Z}\otimes_{\mathbb{Z}G} P_{g(n-1)} & \to & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow 1\otimes\partial_2 & & \downarrow 1\otimes\partial'_2 & & \downarrow 1\otimes\partial''_2 & & \\
0 & \to & \mathbb{Z}\otimes_{\mathbb{Z}G} P_{k1} & \to & \mathbb{Z}\otimes_{\mathbb{Z}G} P_{h1} & \to & \mathbb{Z}\otimes_{\mathbb{Z}G} P_{g1} & \to & 0 \\
 & & \downarrow 1\otimes\partial_1 & & \downarrow 1\otimes\partial'_1 & & \downarrow 1\otimes\partial''_1 & & \\
0 & \to & \mathbb{Z}\otimes_{\mathbb{Z}G} P_{k0} & \to & \mathbb{Z}\otimes_{\mathbb{Z}G} P_{h0} & \to & \mathbb{Z}\otimes_{\mathbb{Z}G} P_{g0} & \to & 0 \\
 & & \downarrow 1\otimes\epsilon & & \downarrow 1\otimes\epsilon' & & \downarrow 1\otimes\epsilon'' & & \\
0 & \to & \mathbb{Z}\otimes_{\mathbb{Z}G} K & \to & \mathbb{Z}\otimes_{\mathbb{Z}G} H & \to & \mathbb{Z}\otimes_{\mathbb{Z}G} G & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3}
$$

Forming

$$H_*(\mathbb{Z}\otimes_{\mathbb{Z}G} P(K)) = \{H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} P(K)\}_{n\geq 0}$$

$$H_*(\mathbb{Z}\otimes_{\mathbb{Z}G} P(H)) = \{H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} P(H)\}_{n\geq 0}$$

$$H_*(\mathbb{Z}\otimes_{\mathbb{Z}G} P(G)) = \{H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} P(G)\}_{n\geq 0}$$

and suppressing K, H and G in their respective projective resolutions we then obtain from(3) the following:

$$
\begin{array}{ccccccccc}
0 & \to & H_*(\mathbb{Z}\otimes_{\mathbb{Z}G} P_K) & \to & H_*(\mathbb{Z}\otimes_{\mathbb{Z}G} P_H) & \to & H_*(\mathbb{Z}\otimes_{\mathbb{Z}G} P_G) & \to & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} P_K) & \to & H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} P_H) & \to & H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} P_G) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_{n-1}(\mathbb{Z}\otimes_{\mathbb{Z}G} P_K) & \to & H_{n-1}(\mathbb{Z}\otimes_{\mathbb{Z}G} P_H) & \to & H_{n-1}(\mathbb{Z}\otimes_{\mathbb{Z}G} P_G) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_1(\mathbb{Z}\otimes_{\mathbb{Z}G} P_K) & \to & H_1(\mathbb{Z}\otimes_{\mathbb{Z}G} P_H) & \to & H_1(\mathbb{Z}\otimes_{\mathbb{Z}G} P_G) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_0(\mathbb{Z}\otimes_{\mathbb{Z}G} P_K) & \to & H_0(\mathbb{Z}\otimes_{\mathbb{Z}G} P_H) & \to & H_0(\mathbb{Z}\otimes_{\mathbb{Z}G} P_G) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{4}
$$

By theorem 2.3 of [2] we obtain from (4) the following long exact sequence:

$$\cdots \longrightarrow H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} P_K) \longrightarrow H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} P_H) \longrightarrow H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} P_G) \xrightarrow{K_n}$$

$$\xrightarrow{k_n} H_{n-1}(\mathbb{Z}\otimes_{\mathbb{Z}G} P_K) \longrightarrow H_{n-1}(\mathbb{Z}\otimes_{\mathbb{Z}G} P_H) \longrightarrow H_{n-1}(\mathbb{Z}\otimes_{\mathbb{Z}G} P_G) \xrightarrow{k_n} \cdots$$

$$\cdots \xrightarrow{k_2} H_1(\mathbb{Z}\otimes_{\mathbb{Z}G} P_K) \longrightarrow H_1(\mathbb{Z}\otimes_{\mathbb{Z}G} P_H) \longrightarrow H_1(\mathbb{Z}\otimes_{\mathbb{Z}G} P_G) \xrightarrow{k_1}$$

$$\xrightarrow{k_1} H_0(\mathbb{Z}\otimes_{\mathbb{Z}G} P_K) \longrightarrow H_0(\mathbb{Z}\otimes_{\mathbb{Z}G} P_H) \longrightarrow H_0(\mathbb{Z}\otimes_{\mathbb{Z}G} P_G) \longrightarrow 0$$

and defining

$$Tor_n^{\mathbb{Z}G}(\mathbb{Z}, K) = H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} P_K)$$

$$Tor_n^{\mathbb{Z}G}(\mathbb{Z}, H) = H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} P_H)$$

$$Tor_n^{\mathbb{Z}G}(\mathbb{Z}, G) = H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} P_G)$$

we get:

$$\cdots \longrightarrow Tor_n^{\mathbb{Z}G}(\mathbb{Z}, K) \longrightarrow Tor_n^{\mathbb{Z}G}(\mathbb{Z}, H) \longrightarrow Tor_n^{\mathbb{Z}G}(\mathbb{Z}, G) \longrightarrow$$

$$\longrightarrow Tor_{n-1}^{\mathbb{Z}G}(\mathbb{Z}, K) \longrightarrow Tor_{n-1}^{\mathbb{Z}G}(\mathbb{Z}, H) \longrightarrow Tor_n^{\mathbb{Z}G}(\mathbb{Z}, G) \longrightarrow \cdots$$

$$\cdots \longrightarrow Tor_1^{\mathbb{Z}G}(\mathbb{Z}, K) \longrightarrow Tor_1^{\mathbb{Z}G}(\mathbb{Z}, H) \longrightarrow Tor_1^{\mathbb{Z}G}(\mathbb{Z}, G) \longrightarrow$$

$$\longrightarrow Tor_0^{\mathbb{Z}G}(\mathbb{Z}, K) \longrightarrow Tor_0^{\mathbb{Z}G}(\mathbb{Z}, H) \longrightarrow Tor_0^{\mathbb{Z}G}(\mathbb{Z}, G) \longrightarrow 0.$$

Putting

$$H_n(G, K) = Tor_n^{\mathbb{Z}G}(\mathbb{Z}, K)$$

$$H_n(G, H) = Tor_n^{\mathbb{Z}G}(\mathbb{Z}, H)$$

$$H_n(G; G) = Tor_n^{\mathbb{Z}G}(\mathbb{Z}, G)$$

etc.

we get:

$$\cdots \longrightarrow H_n(G, K) \longrightarrow H_n(G, H) \longrightarrow H_n(G, G) \longrightarrow$$

$$\longrightarrow H_{n-1}(G, K) \longrightarrow H_{n-1}(G, H) \longrightarrow H_{n-1}(G, G) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_1(G, K) \longrightarrow H_1(G, H) \longrightarrow H_1(G, G) \longrightarrow$$

$$\longrightarrow H_0(G, K) \longrightarrow H_0(G, H) \longrightarrow H_0(G, G) \longrightarrow 0$$

Since $G = \mathbb{Z}G \otimes_{\mathbb{Z}} B$ it follows from page 96 of [1] that
$H_n(G, G) = Tor_n^{\mathbb{Z}G}(G, G) = Tor_n^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G \otimes_{\mathbb{Z}} B) = 0$

we then obtain short exact sequences

$$0 \longrightarrow H_n(G, K) \longrightarrow H_n(G, H) \longrightarrow 0$$

$$0 \longrightarrow H_{n-1}(G, K) \longrightarrow H_{n-1}(G, H) \longrightarrow 0$$

$$\cdots$$

$$0 \longrightarrow H_1(G, K) \longrightarrow H_1(G, H) \longrightarrow 0$$

Hence
$H_n(G, K) \cong H_n(G, H) \quad \forall \quad n \geq 1.$

References

[1] P.M. Cohn, *Further Algebra and Applications,* Springer-Verlag, London, 2003.

[2] E. Lluis-Puebla, *Higher K - Theory: an overview IX. Introduction to Algebraic K - Theory. Proceedings of the first seminar on algebraic K - theory at the Universidad Nacional Autonoma de Mexico in 1985,* Springer-Verlag, Berlin, 1992.

[3] W.H. Mannan, Periodic Cohomology, *Homology, Homotopy and Applications,* **10(2)** (2008), 135 - 137.

Received: January 4, 2013