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# Classifying fuzzy subgroups of finite dihedral group $D_{2p^s}$

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**ABSTRACT**

In this paper, we give an explicit formula for counting the number of distinct fuzzy subgroups of dihedral group $D_{2p^s}$ for any prime(p) and any integer $s \geq 1$. This we achieved using the algorithm described on the most recent equivalence relation $\approx$ known in the literature to classify fuzzy groups.

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## 1. Introduction

In Mathematics, it is more convenient to study the relationship between objects rather than studying individual objects as members of a set. This leads to the formulation of the concept of relations sometimes called correspondences. Generally, a relation describes a connection between two or more things. Relation is usually studied between two or more non-empty sets. Since a set is specified by its elements, a relation is usually described between two elements of a given set.

To classify the elements of a given set is basically to divide the set into disjoint subsets usually called classes. This concept leads to the concept of equivalence relation on the set. This notion generalizes equality in the sense that objects or elements that are related or equal in some sense are classified in the same class.

Research had focused on classification of distinct fuzzy subgroups of finite dihedral groups $G$, with respect to the natural equivalence relation $\sim$. Thus, an explicit formula for counting the number of distinct fuzzy subgroups of dihedral groups $G$ of the form $D_{2p^s}$ (where $p$ is a prime and $s$ is a positive integer) had been derived with respect to $\sim$, see Tarnauceanu (2012). However, this study was designed to extend the results in Olayiwola and Isyaku (2018) and Olayiwola and Garba (2018) by establishing explicit formula for counting the number of distinct fuzzy subgroups of dihedral groups $D_{2p^s}$ with respect to the equivalence relation $\approx$ defined on the lattice $(FL(D_{2p^s}))$ of all fuzzy subgroups of the group $D_{2p^s}$.

The abstract description of $D_{2p^s}$ was obtained by using its generators. The distinct equivalence classes of $FL(D_{2p^s})$ modulo $\approx$ was obtained by using modified Burnside's lemma and the action $\rho$ of automorphism group $(Aut(D_{2p^s}))$ on $FL(D_{2p^s})$. The

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action $\rho$ was seen as an action on all chains $\overline{\mathcal{C}}$ of subgroups of $G$ terminated in $G$. An equivalence relation $\approx$ was defined on $\overline{\mathcal{C}}$. By using basic elementary results of group theory, the subgroup lattice of $D_{2p^s}$ and its subsets belonging to the set that are fixed by each automorphism $(Fix(f))$ was determined. The value of $|Fix_{\overline{C}}(f)|$ was obtained by computing the number of chains of subgroup lattice of $D_{2p^s}$ which ends in $D_{2p^s}$. Computing the number of distinct fuzzy subgroups of $D_{2p^s}$ was obtained by observing several patterns of chains of subgroups lattices of $D_{2p^s}$ that were fixed by $Aut(D_{2p^s})$, and the formula $N=\frac{1}{|Aut(D_{2p^s})|}\sum_{f\in Aut(D_{2p^s})}|fix_{\overline{C}}(f)|$. Several values were obtained from different computations and explicit formulas for computing the number of fuzzy subgroups of $G$ was derived.

For other notion of equivalence relations of fuzzy sets used to study equivalence of fuzzy subgroups, see Sulaiman (2012), Murali and Makamba (2001, 2003), and Ndiweni (2014).

## 2. Method

In this section, we give some definitions, preliminary results, and the equivalence relations introduced in Tarnauceanu (2016).

Definition 1: Given an arbitrary non-empty set $X$, a fuzzy set $(on\ X)$ is a function from $X$ to the unit interval $I:[0,1]$. That is $\mu : X \mapsto I$. Let $G$ be a group and $\mathcal{F}(G)$ be a collection of all fuzzy subsets of $G$. An element $\mu$ of $\mathcal{F}(G)$ is said to be a fuzzy subgroup of $G$ if it satisfies the following two conditions:

{i} $\mu(xy) \geq min\{\mu(x),\mu(y)\}, \forall x,y \in G.$

{ii} $\mu(x^{-1}) \geq \mu(x), \forall x \in G.$

In this situation, we have $\mu(x^{-1}) = \mu(x),\ for\ any\ x \in G,\ and\ \mu(e) = max\mu(G) = \sup\mu(G).$

The set $FL(G)$ which consist of all the fuzzy subgroups of $G$ forms a lattice with respect to the usual ordering of fuzzy set called fuzzy subgroup lattice.

For any $\alpha \in [0,1]$, the level subset is defined by $\mu_\alpha = \{x \in G \mid \mu(x) \geq \alpha\}$ (). Thus, a fuzzy subset $\mu$ is a fuzzy subgroup of $G$ *iff* its level subsets are subgroups of $G$.

The fuzzy subgroups of $G$ had been classified up to some natural equivalence relations on $\mathcal{F}(G)$ as follows. Let $\mu,\eta \in \mathcal{F}(G)$ then $\mu \sim \eta$ *iff* $(\mu(x) > \mu(y) \Leftrightarrow \eta(x) > \eta(y) \forall x,y \in G)$ and two fuzzy subgroups $\mu,\eta$ of $G$ are said to be distinct if $\mu \nsim \eta$.

This concept has also been linked to level subgroups as follows (). Suppose $\mu(G) = \{\alpha_1,\alpha_2,\cdots,\alpha_r\}$ and assume that $\alpha_1 > \alpha_2 > \cdots > \alpha_r$. Then, $\mu$ determines a chain of subgroups of $G$ which ends in $G$, that is $\mu G_{\alpha_1} \subset \mu G_{\alpha_2} \subset \cdots \subset \mu G_{\alpha_r} = G$ (1)

Relative to the level subgroups, two fuzzy subgroups $\mu\ and\ \eta$ are said to belong to the same equivalence class *iff* they have the same set of level subgroups, that is, they determine a chain of type (1). Thus, counting all distinct fuzzy subgroups of $G$ is reduced to counting the chains of subgroups of $G$ that ends in $G$.

Definition 2.1: Let $\Omega$ be an arbitrary non-empty set and $G$ be a group. An action or operator $\rho$ of $G$ on $\Omega$ is a map $\Omega \times G \mapsto \Omega$ satisfying the following axioms.

{i.} $\rho(\omega, g_1, g_2) = \rho(\rho(\omega, g_1), g_2),$
$\forall g_1, g_2 \in G\ and\ \omega \in \Omega;$

{ii.} $\rho(\omega, e) = \omega, \forall \omega \in \Omega;$

Groups actions generalizes group multiplication. If $G$ acts or operates on $\Omega$ so does any subgroup of $G$. $\Omega$ is called a $G$ -set. Note that we do not require $\Omega$ to be related to $G$ in anyway. However, group actions are more interesting if the set $\Omega$ is somehow related to the group $G$.

It is well-known that every group action $\rho$ of $G$ induces an equivalence relation $R_\rho$ on $\Omega$ defined by $\omega R_\rho \nu$ *iff* $\exists g \in G$ such that $\nu = \rho(\omega, g)$. The quotient set with respect to action $\rho$ is called the orbits of $\Omega$. Let $Fix_\Omega(g) = \{\omega \in \Omega | \rho(\omega, g) = \omega\}$ be the set of elements of $\Omega$ that are fixed by $g$. If both $G$ and $\Omega$ are finite, then the number of distinct orbits of $\Omega$ relative to $\rho$ $\rho$ is given by $N = \frac{1}{|G|}\sum_{g \in G}|Fix_\Omega(g)|$, known as Burnside's Lemma. The results will be applied in counting the number of distinct fuzzy subgroups of $G$ with respect to the new equivalence relation on $FL(G)$, induced by the action of automorphism group $Aut(G)$ associated to $G$ on $FL(G)$.

In the following, we give an overview of the new equivalence relations defined in Tarnauceanu (2016).

Let be a finite group, then the action $Aut(G)$ on $FL(G)$ given by $\rho : FL(G) \times Aut(G) \mapsto FL(G)$, $\rho : (\mu, f) = \mu f, \forall (\mu, f) \in FL(G) \times Aut(G)$ is clearly well-defined. It is well-known that every group action $\rho$ induces an equivalence relation, let $\approx_\rho$ be equivalence relation on $FL(G)$ induced by the action of $\rho$ of $Aut(G)$, that is $\mu \approx_\rho \eta,\ iff\ \exists f \in Aut(G)\ such\ that\ \eta = \mu$. The action $\rho$ defined above can be seen in terms of chains of subgroups of $G$ since every fuzzy subgroup determine a chain of subgroups of $G$ which end in $G$. The concentration can now be shifted to chains of subgroups by considering the equivalence relation $\approx$ on $FL(G)$ defined by $\eta \approx \mu \ iff \ \exists f \in Aut(G) \ni f(\mathcal{C}_\eta) = \mathcal{C}_\mu$. Where $\mathcal{C}_\mu : \mu G_{\alpha_1} \subset \mu G_{\alpha_2} \subset \cdots \subset \mu G_{\alpha_m} = G$ and $\mathcal{C}_\eta : \eta G_{\beta_1} \subset \eta G_{\beta_2} \subset \cdots \subset \eta G_{\beta_n} = G$. Now to compute the number of distinct fuzzy subgroups $N$ of $G$ with respect to $\approx$, let $\overline{\mathcal{C}}$ be the set consisting of all chains of subgroups of $G$ terminated in $G$. Then the previous action $\rho$ of $Aut(G)$ on $FL(G)$ can be seen as an action of $Aut(G)$ on $\overline{\mathcal{C}}$ and $\approx_\rho$ as the equivalence relation induced by the action. A similar equivalence relation to $\approx$ can be constructed on $\overline{\mathcal{C}}$ as follows; For two chains, $\mathcal{C}_1 : H_1 \subset H_2 \subset \cdots \subset H_m = G$ and $\mathcal{C}_2 : K_1 \subset K_2 \subset \cdots \subset K_n = G$ of $\overline{\mathcal{C}}$, we set $\mathcal{C}_1 \approx \mathcal{C}_2 \ iff \ m = n$ and $\exists f \in Aut(G) \ni f(H_i) = K_i$ $\forall i = 1, 2, \cdots, n$. Here, the orbit of a chain $\mathcal{C} \in \overline{\mathcal{C}}$ is the set $\{f(\mathcal{C}) \mid f \in Aut(G)\}$, and the set of all chains $\mathcal{C}$ that are fixed by an automorphism $f \ of \ G$ is $Fix_{\overline{\mathcal{C}}}(f) = \{\mathcal{C} \in \overline{\mathcal{C}} \mid f(\mathcal{C}) = \mathcal{C}\}$. From Burnside's Lemma, the number $N$ of distinct fuzzy subgroups of $G$ is given by $N = \dfrac{1}{|\,Aut(G)\,|} \sum_{f \in Aut(G)} |Fix_{\overline{\mathcal{C}}}(f)|$, and we remark that the formula above can be used to calculate the number ($N$) of fuzzy subgroups of any finite group provided the lattice subgroup $L(G)$ and the automorphism group $Aut(G)$ of the group are known.

## 3. Results

### *3.1. The number of fuzzy subgroups of finite dihedral group D$_{\acute{C}p^s}$*

Dihedral groups of order $2n$, that is $D_{2n}$ for $n \geq 2$ is generated by two involutions say; $a$ of order $n$ and b of order 2. This can be presented as $D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle$. In the literature, the automorphism group of $D_{2n}$ is also well-known and is given by

$$Aut(D_{2n}) = \{f_{\alpha,\beta} \mid \alpha = \overline{0, n-1} \ with \ (\alpha, n) = 1, \beta = \overline{0, n-1}\}$$

where $f_{\alpha,\beta} : D_{2n} \mapsto D_{2n}$ *is defined by* $f_{\alpha,\beta}(a^i) = a^{\alpha i}$ and $f_{\alpha,\beta}(a^i b) = a^{\alpha i + \beta} b, \ \forall \ i = \overline{0, n-1}$.

In Tarnauceanu (2016), the order of automorphism group of $D_{2n}$ was given by $|\,Aut(D_{2n})\,| = |\,n\phi(n)\,|$. We remark that this is only true for $n$ greater two. The structure of the subgroup lattice $L(D_{2n})$ of $D_{2n}$ is also known in the literature: for every divisor $r$ or $n$, $D_{2n}$ possesses a subgroup isomorphic to $\mathbb{Z}_r$ namely $H_0^r = \langle a^{\frac{n}{r}} \rangle$, *and* $\frac{n}{r}$ subgroups isomorphic to $D_r$, given by $H_i^r = \langle a^{\frac{n}{r}}, a^{i-1} b \rangle$, $i = 1, 2, \cdots, \frac{n}{r}$. Now for each $f_{\alpha,\beta} \in Aut(D_{2n})$, let $Fix(f_{\alpha,\beta})$ be the set consisting of all subgroups of $D_{2n}$ that are invariant relative to $f_{\alpha,\beta}$, that is $Fix(f_{\alpha,\beta}) = \{H \leq D_{2n} \mid f_{\alpha,\beta}(H) = H\}$.

The subgroup of type $H_0^r$ belongs to $Fix(f_{\alpha,\beta}) \ iff \ (\alpha, r) = 1$, while a subgroup of type $H_i^r$ belongs to $Fix(f_{\alpha,\beta}) \ iff \ (\alpha, r) = 1$ and $\frac{n}{r}$ *divides* $(\alpha - 1)(i - 1) + \beta$. Now computing $|\,Fix_{\overline{\mathcal{C}}} f_{\alpha,\beta}\,|$ implies computing the number of chains of $L(D_{2n})$ which end in $D_{2n}$ and are contained in the set $Fix(f_{\alpha,\beta})$.

Next, we characterize all equivalent fuzzy subgroups of $D_{2p^s}$. We consider some cases and apply the result obtained in the generalization through proving by induction.

**Case** $s = 1, p = 3$

In this case, we have $D_{2(3)} = D_6$, from the algorithm defined above and lattice subgroup structure, we have; $Aut(D_6) = \{f_{1,0}, f_{1,1}, f_{1,2}, f_{2,0}, f_{2,1}, f_{2,2}\}$ and the subgroups of $D_6$ are as follows; $H_0^1 = \langle e \rangle, H_0^3 = \langle a \rangle, H_1^3 = \langle a, b \rangle, H_n^1 = \langle a^{n-1} b \rangle \ for \ n = 1, 2, 3$.

**Table 1.** Subgroups fixed by $Aut(D_6)$.

| $\alpha$ | $\beta$ | $Fix(f_{\alpha,\beta})$ |
|---|---|---|
| 1 | 0 | $L(D_6)$ |
| 1 | 1,2 | $\{H_0^1, H_0^3, H_1^3\}$ |
| 2 | 0 | $\{H_0^1, H_0^3, H_1^1, H_1^3\}$ |
| 2 | 1 | $\{H_0^1, H_0^3, H_3^1, H_1^3\}$ |
| 2 | 2 | $\{H_0^1, H_0^3, H_2^1, H_1^3\}$ |

Now the number $N$ of distinct fuzzy subgroups of $D_6$ with respect to $\approx$ is given by

$$N = \frac{1}{6}[10 + (4 * 2) + 6 + 6 + 6] = 6.$$

**Case** $s = 1, \ p = 5$

In this case, we have $D_{2*5} = D_{10}$ and $Aut(D_{10}) = \{f_{1,0}, f_{1,1}, f_{1,2}, f_{1,3}, f_{1,4}, f_{2,0}, f_{2,1}, f_{2,2}, f_{2,3}, f_{2,4}, f_{3,0}, f_{3,1}, f_{3,2}, f_{3,3}, f_{3,4}, f_{4,0}, f_{4,1}, f_{4,2}, f_{4,3}, f_{4,4},\}$ while the subgroups of $D_{10}$ are given by $H_0^1 = \langle e \rangle$, $H_n^1 = \langle a^{n-1} b \rangle \ for \ n = 1, 2, 3, 4, 5.$, $H_1^5 = \langle a, b \rangle$ and $H_0^5 = \langle a \rangle$.

Now the number $N$ of distinct fuzzy subgroups of $D_{10}$ with respect to $\approx$ is given by

**Table 2.** Number of chains of subgroups fixed by $Aut(D_6)$.

| $\lvert Fix_{\bar{C}}(f_{1,0})\rvert$ | $\lvert Fix_{\bar{C}}(f_{1,n})\rvert$ *for* $n=1,2$ | $\lvert Fix_{\bar{C}}(f_{2,n})\rvert$ *for* $n=0,1,2.$ |
|---|---|---|
| 10 | 4 | 6 |

**Table 3.** Subgroups fixed by $Aut(D_{10})$.

| $\alpha$ | $\beta$ | $Fix(f_{\alpha,\beta})$ |
|---|---|---|
| 1 | 0 | $L(D_{10})$ |
| 1 | 1,2,3,4 | $\{H_0^1, H_0^5, H_1^5\}$ |
| 2,3,4 | 0,0,0 | $\{H_0^1, H_0^5, H_1^1, H_1^5\}$ |
| 2,4 | 1,3 | $\{H_0^1, H_0^5, H_5^1, H_1^5\}$ |
| 2,3,4 | 2,4,1 | $\{H_0^1, H_0^5, H_4^1, H_1^5\}$ |
| 2,3,4 | 3,2,4 | $\{H_0^1, H_0^5, H_3^1, H_1^5\}$ |
| 2,3,4 | 4,3,2 | $\{H_0^1, H_0^5, H_2^1, H_1^5\}$ |

**Table 4.** Number of chains of subgroups fixed by $Aut(D_{10})$.

| $\lvert Fix_{\bar{C}}(f_{1,0})\rvert$ | $\lvert Fix_{\bar{C}}(f_{1,n})\rvert$ *for* $n=1,2,3,4$ | $\lvert Fix_{\bar{C}}(f_{m,n})\rvert$ *for* $m=2,3,4$ *and* $n=0,1,2,3,4$ |
|---|---|---|
| 14 | 4 | 6 |

$$N=\frac{1}{20}[14+(4*4)+(6*3)5]=6.$$

Similarly, for other primes, we obtained the same result, thus we have the following;

### *3.1. Lemma*

Let $G$ be a dihedral group of the form $G=D_{2p^s}$, where $p$ is any prime number greater than 2 and s=1. Then, the number of distinct fuzzy subgroups of G with respect to $\approx$ is 6.

**Proof:**

Observe that the sets of subgroups of $D_{2p^s}$ that are fixed $(Fix(f_{\alpha,\beta}))$ by elements of the automorphism groups of $D_{2p^s}$ can take either of the following forms; $\{L(D_{2p^s})\}$ or $\{H_0^1, H_0^p, H_1^p\}$ or $\{H_0^1, H_0^p, H_1^p, H_k^1\}$, for $(k,p)=1$ *and* $k\le p$. Corresponding to each set is a value $\lvert Fix_{\bar{C}}(f_{\alpha,\beta})\rvert$, which is the number of chains of $L\left(D_{2p^s}\right)$ which ends in $D_{2p^s}$ and are contained in $Fix(f_{\alpha,\beta})$. In particular, we have

$L\left(D_{2p^s}\right)$ that corresponds to the value $\lvert Fix_{\bar{C}}(f_{1,0})\rvert = \frac{2}{p-1}(p^2+p-2)$ by Tarnauceanu (2012).

$\{H_0^1, H_0^p, H_1^p\}$ corresponds to the value $\lvert Fix_{\bar{C}}(f_{1,\beta})\rvert = 4$ *for any* $\beta=\overline{1,p}$.

$\{H_0^1, H_0^p, H_1^p, H_k^1\}$ corresponds to the value $\lvert Fix_{\bar{C}}(f_{\alpha,\beta})\rvert = 6$ for any $\beta=\overline{0,p}$ with $(\alpha,p)=1,\ \ \alpha>1$.

Note that for any $p$ the subgroups $\{H_0^1, H_0^p, H_1^p\}$ appears $4(p-1)$ times, thus $\lvert Fix_{\bar{C}}(f_{1,\beta})\rvert = 4(p\text{-}1)$ *for any* $\beta=\overline{1,p}$.

The subgroups $\{H_0^1, H_0^p, H_1^p, H_k^1\}$ appears $6p$ $6p(p-2)$ times, thus $\lvert Fix_{\bar{C}}(f_{\alpha,\beta})\rvert = 6p(p-2)$ *for any* $\beta=\overline{1,p}$.

Hence, $N=\frac{1}{\lvert Aut(G)\rvert}\sum_{f\in Aut(G)}\lvert Fix_{\bar{C}}(f_{\alpha,\beta})\rvert = \frac{6(p-1)}{\phi(p)} = 6$ □.

**Table 5.** Subgroups fixed by $Aut(D_{18})$.

| $\alpha$ | $\beta$ | $Fix(f_{\alpha,\beta})$ |
|---|---|---|
| 1 | 0 | $L(D_{18})$ |
| (1)(4)(7) | (1,2,4,5,7,8)(1,2,4,5,7,8)(1,2,4,5,7,8) | $\{H_0^1, H_0^3, H_0^9, H_1^9\}$ |
| 1 | 3,6 | $\{H_0^1, H_0^3, H_0^9, H_1^9, H_1^3, H_2^3, H_3^3\}$ |
| 2,5,8 | 0 | $\{H_0^1, H_0^3, H_0^9, H_1^9, H_1^1, H_1^3\}$ |
| 2,5,8 | 1,4,7 | $\{H_0^1, H_0^3, H_0^9, H_1^9, H_9^1, H_3^3\}$ |
| 2,5,8 | 2,8,5 | $\{H_0^1, H_0^3, H_0^9, H_1^9, H_8^1, H_2^3\}$ |
| 2,5,8 | 3 | $\{H_0^1, H_0^3, H_0^9, H_1^9, H_7^1, H_1^3\}$ |
| 2,5,8 | 4,7,1 | $\{H_0^1, H_0^3, H_0^9, H_1^9, H_6^1, H_3^3\}$ |
| 2,5,8 | 5,2,8 | $\{H_0^1, H_0^3, H_0^9, H_1^9, H_5^1, H_2^3\}$ |
| 2,5,8 | 6 | $\{H_0^1, H_0^3, H_0^9, H_1^9, H_4^1, H_1^3\}$ |
| 2,5,8 | 7,1,4 | $\{H_0^1, H_0^3, H_0^9, H_1^9, H_3^1, H_3^3\}$ |
| 2,5,8 | 8,5,2 | $\{H_0^1, H_0^3, H_0^9, H_1^9, H_2^1, H_2^3\}$ |
| 4,7 | 0 | $\{H_0^1, H_0^3, H_0^9, H_1^9, H_1^1, H_4^1, H_7^1, H_1^3, H_2^3, H_3^3\}$ |
| 4,7 | 3,6 | $\{H_0^1, H_0^3, H_0^9, H_1^9, H_3^1, H_6^1, H_9^1, H_1^3, H_2^3, H_3^3\}$ |
| 4,7 | 6,3 | $\{H_0^1, H_0^3, H_0^9, H_1^9, H_2^1, H_5^1, H_8^1, H_1^3, H_2^3, H_3^3\}$ |

**Table 6.** Number of chains of subgroups fixed by $Aut(D_{18})$.

| $\lvert Fix_{\bar{C}}(f_{1,0})\rvert$ | $\lvert Fix_{\bar{C}}(f_{1,n})\rvert$ for $n = 3,6$ | $\lvert Fix_{\bar{C}}(f_{m,n})\rvert$ for $m = 2,5,8$ and $n = 0,1,2,3,4,5,6,7,8$ | $\lvert Fix_{\bar{C}}(f_{m,n})\rvert$ for $m = 1,4,7$ and $n = 1,2,4,5,7,8$ | $\lvert Fix_{\bar{C}}(f_{m,n})\rvert$ for $m = 4,7$ and $n = 0,3,6$ |
|---|---|---|---|---|
| 56 | 20 | 16 | 8 | 32 |

**Case** $s = 2$ and $p = 3$

Observe that the case $p$ = 2 and $s$ = 2, that is, $D_{2(2)^2} = D_8$, had been resolved (see Tarnauceanu 2016) where $N$ = 16 was obtained as the number of distinct fuzzy subgroups of $D_8$. We now proceed to the case where $s = 2$ and $p = 3$ that is, $D_{2(3)^2} = D_{18}$.

$$Aut(D_{18}) = \{f_{1,0}, f_{1,1},, f_{1,2}, f_{1,3}, f_{1,4}, f_{1,5}, f_{1,6}, f_{1,7}, f_{1,8}, f_{2,0}, f_{2,1}, f_{2,2}, f_{2,3}, f_{2,4}, f_{2,5}, f_{2,6}, f_{2,7}, f_{2,8},, f_{4,0}, f_{4,1}, f_{4,2}, f_{4,3}, f_{4,4}, f_{4,5}, f_{4,6}, f_{4,7}, f_{4,8}, f_{5,0}, f_{5,1}, f_{5,2}, f_{5,3}, f_{5,4}, f_{5,5}, f_{5,6}, f_{5,7}, f_{5,8},, f_{7,1}, f_{7,2}, f_{7,3}, f_{7,4}, f_{7,5}, f_{7,6}, f_{7,7}, f_{7,8}\, f_{8,1}, f_{8,2}, f_{8,3}, f_{8,4}, f_{8,5}, f_{8,6}, f_{8,7}, f_{8,8}\}$$

Now, the number $N$ of distinct fuzzy subgroups of $D_{18}$ is given by

$$N = \frac{1}{54}[56 + (18*8) + (20*2) + (16*27) + (32*6)] = 16.$$

Similarly, we obtained 16 distinct fuzzy subgroups for $D_{2(5)^2} = D_{50}, D_{2(7)^2} = D_{98}$.

As a Lemma to our main result we have the following;

### 3.2. Lemma

Let $G$ be a dihedral group of the form $D_{2p^2}$. Then, the number of distinct fuzzy subgroups of G with respect to $\approx$ is 16.

**Proof:** The proof is as Lemma 3.1. □.

For $s = 3$, we have $D_{2p^3}$. Using a similar approach, we found that the number of distinct fuzzy subgroups of $D_{2p^3}$ with respect to $\approx$ is 40.

More generally, we have the following result;

### 3.3. Theorem

For any $s \geq 1, p \succ 2$ the number $N$ of all distinct fuzzy subgroups of $D_{2p^s}$ with respect to $\approx$ is given by $(s+2)2^s$.

**Proof:**

We assume without loss of generality that $p = 3$. We observe that there is a one-to-one correspondence between the number of distinct fuzzy subgroups of $D_{2(3)^s}$ and the order of $|Fix_{\bar{C}}(f_{2,\beta})|$ for any $\beta = \overline{\{0, 3^s - 1\}}$.

We now prove by induction. The statement is clearly true for $s =$ 1,2, and 3 as shown in our computation above.

Assume that the statement is true for $s = k$, that is, $D_{2(3)^k}$ has $(k+2)2^k$ distinct fuzzy subgroups then set $\beta = 0$, so the following subgroups are fixed for the case $s = k$, that is, $Fix(f_{2,0})_k = \{H_0^1, H_1^1, H_0^3, H_1^3, H_0^{3^2}, H_1^{3^k}, H_1^{3^k}, H_1^{3^2}, \cdots, H_1^{3^k}, H_0^{3^k}\}$, which in turn gives the corresponding value $|Fix_{\bar{C}}(f_{2,0})|_k = (k+2)2^k$. We now show that $D_{2(3)^{k+1}}$ has $[(k+1)+2)]2^{k+1}$ distinct fuzzy subgroups. The subgroups that are fixed by the element of automorphism group $f_{2,0}$ for the case $s = k+1$ are given $Fix(f_{2,0})_{k+1} = \{H_0^1, H_1^1, H_0^3, H_1^3, H_0^{3^2}, H_1^{3^2}, \cdots, H_1^{3^k}, H_1^{3^k}, H_1^{3^{k+1}}, H_0^{3^{k+1}}\}$, this is evident from the structure of subgroup lattice of $D_{2(3)^{k+1}}$. Note that the set $Fix(f_{2,0})_{k+1}$ has two more subgroups than $Fix(f_{2,0})_k$. From the structure of lattice subgroups of $D_{2(3)^{k+1}}$, these two subgroups yield a further $2^k(k+4)$ distinct fuzzy subgroups. Now adding together, we have the total number of distinct fuzzy subgroups of $D_{2p^{k+1}}$,

$$N = |Fix_{\bar{C}}(f_{2,0})| = (k+2)2^k + 2^k(k+4) = [(k+1)+2]2^{k+1}.$$

Hence, this completes the induction □.

## 4. Discussion

We remark that our classification excludes $D_2$ and $D_4$. In other words, $D_2$ and $D_4$ are the two Abelian examples of dihedral groups known in the literature. Although they can be considered as dihedral groups of order 2 and 4, respectively, because they satisfy the relations of being dihedral, though their geometric interpretations slightly differ.

## 5. Conclusion

This study has solved the problem for counting the number of fuzzy subgroups of finite dihedral groups $D_{2p^s}$ with respect to the equivalence relation $\approx$. The counting technique used is very broad and requires a lot patience and concentration. As the order of the group gets increasingly large, constructing the subgroups lattices becomes difficult. However, our formula has reduced extensively the rigorousness.

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