International Journal of Pure and Applied Mathematics
Volume 116 No. 2 2017, 313-328
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v116i2.3
ijpam.eu

POSITIVE SOLUTIONS TO A NONLINEAR EIGENVALUE PROBLEM OF FRACTIONAL DIFFERENTIAL EQUATION WITH INTEGRO-DIFFERENTIAL BOUNDARY CONDITIONS

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Abstract: In this paper, we establish the existence of positive solutions for eigenvalue problem of fractional differential equations with integro-differential boundary conditions. We determine the intervals of parameter λ for which the existence of positive solutions is guaranteed. An example is also presented to show the application of our results.

AMS Subject Classification: 34A08, 34B09, 34B18
Key Words: eigenvalue, positive solutions, fractional derivative, integro-differential boundary conditions, cones

1. Introduction

In this paper, we discuss the existence of positive solutions to the following eigenvalue problem of fractional differential equation:

$$\left.\begin{aligned} D^\alpha u(t) + \lambda w(t) f(t,u(t)) = 0, \;\; t \in (0,1), \\ u(0)=0, \;\; u(1) + D^r u(1) = \int_0^1 p(s)u(s)ds, \end{aligned}\right\} \tag{1.1}$$

where $1 < \alpha \leq 2,\; 0 < r < 1,\; D^\alpha$ and D^r are the standard Riemann-Liouville fractional derivatives, $w \in C([0,1],\; I\!R^+),\;\; f \in C([0,1] \times I\!R^+, I\!R^+),\;\; p \in \mathcal{L}^1[0,1]$ is non-negative and λ is a positive parameter.

Due to the rapid development and wide applications of fractional calculus

Received: November 14, 2016
Revised: June 17, 2017
Published: October 7, 2017

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in many fields of engineering and applied sciences, the study of the existence of positive solutions to boundary value problems(BVP) of fractional differential equation has caught the attention of many researchers, see [1], [7], [8], [15], [22], [23], [24], [26], [29], [30] and the references cited therein.

In 1997, J. Henderson and H. Wang[14] established the intervals of the values of parameter λ for which there exist positive solutions to their eigenvalue problem. Since then, many papers have focused in this direction and the study of existence results for eigenvalue problems of fractional differential equations under various boundary conditions has been on the increase, see [3], [4], [9], [10], [11], [12], [13], [15], [16], [20], [25], [27], [28], [31] and the references therein.

In particular, M. El-Shahed in [9] discussed the existence and nonexistence of positive solutions to the following boundary value problem:

$$\left.\begin{aligned} D^{\alpha}u(t)+\lambda a(t)f(u(t))=0,\ \ t\in(0,1),\\ u(0)=u''(0)=0,\ \ \gamma u'(1)+\beta u''(1)=0, \end{aligned}\right\} \tag{1.2}$$

where $2<\alpha<3$, D^{α} is the standard Riemann-Liouville fractional derivative and λ is a positive parameter.

The authors in [31] studied the existence of positive solutions to the following boundary value problem:

$$\left.\begin{aligned} D^{\alpha}u(t)+\lambda h(t)f(u(t))=0,\ \ t\in(0,1),\\ u(0)=u'(0)=0,\ \ u(1)=\int_0^1 g(s)u(s)ds, \end{aligned}\right\} \tag{1.3}$$

where $2<\alpha\le 3$, D^{α} is the standard Riemann-Liouville fractional derivative and λ is a positive parameter.

Inspired by the works in [9] and [31], the aim of this paper is to determine the intervals for the values of parameter λ such that the existence of at least one positive solution to the BVP(1.1) is guaranteed. Here, our boundary condition is allowed to depend on the fractional derivative $D^{r}u(t)$ of the unknown function $u(t)$ which the authors in [3], [4], [9], [11], [12], [13], [14], [20], [25], [31] did not consider. However, to the best of our knowledge, the existence of positive solutions to the BVP(1.1) has not been discussed so far. Our approach is based on the well-known Krasnosel'skii fixed-point theorem in a cone and we have different results.

Throughout this work, we assume the following conditions hold:

C_1. $f:[0,1]\times[0,\infty)\longrightarrow[0,\infty)$ is continuous.

C_2. $w:(0,1)\longrightarrow[0,\infty)$ is continuous with $w(t)\neq 0$ on any closed subinterval of $(0,1)$ and $0<\int_0^1 w(t)dt<\infty$.

C_3. $p:(0,1)\longrightarrow[0,\infty)$ is continuous and $0\leq b_{_o}=\dfrac{1}{a_{_o}}\displaystyle\int_0^1 p(t)t^{\alpha-1}dt<1.$

C_4. $\displaystyle\lim_{u\to 0^+}\frac{f(t,u)}{u}=\mathcal{L}_{_o}$ exists.

C_5. $\displaystyle\lim_{u\to\infty}\frac{f(t,u)}{u}=\mathcal{M}_{_o}$ exists.

The rest of the paper is outlined as follows: In section 2, we recall some basic definitions and some known results. We also present the expression and some basic properties of the integral kernel $G_{_o}(t,s)$. In section 3, our main results are stated and proved. Finally, we give an example in section 4 to illustrate the practical application of our main results.

2. Preliminary Results

In this section, we recall some basic definitions and results. Further, we give the expression and basic properties of the kernel $G_{_o}(t,s)$ associated with the BVP(1.1).

Definition 2.1([6], [19], [21]) - The Riemann-Liouville fractional integral of order $\alpha>0$ for a given continuous function $f:(0,\infty)\longrightarrow I\!R$ is defined to be

$$I^{\alpha}_{0^+}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds,$$

provided the right side is pointwise defined on $(0,\infty)$.

Definition 2.2([6], [19], [21]) - The Riemann-Liouville fractional derivative of order $\alpha>0$ for a given continuous function $f:(0,\infty)\longrightarrow I\!R$ is defined to be

$$D^{\alpha}_{0^+}f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t (t-s)^{n-\alpha-1}f(s)ds,$$

$n-1<\alpha\leq n$, provided the right side is pointwise defined on $(0,\infty)$, where $n=[\alpha]+1$ and $[\alpha]$ is the integer part of the number α.

Remark 2.3 *If* $u\in C(0,1)\cap\mathcal{L}(0,1)$, *then*

$$D^{\alpha}I^{\alpha}u(t)=u(t).$$

Lemma 2.4 *(see [5]) - Let* $\alpha > 0$. *If we assume* $u \in C(0,1) \cap \mathcal{L}(0,1)$, *then the fractional differential equation* $D^{\alpha}u(t) = 0$ *has*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

for $c_i \in I\!R$, $i = 1, 2, \dots n$, *as unique solutions, where* n *is the smallest integer greater than or equal to* α.

Lemma 2.5 *(see [5], [6]) - Assume that* $u \in C(0,1) \cap \mathcal{L}(0,1)$ *with a fractional derivative of order* $\alpha > 0$ *that belongs to* $C(0,1) \cap \mathcal{L}(0,1)$. *Then*

$$\left.\begin{aligned} I^{\alpha}D^{\alpha}u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \\ \text{for } c_i \in I\!R, \quad i = 1, 2, \dots, n. \end{aligned}\right\} \tag{2.6}$$

Lemma 2.7 *(see [18]) - Assume that* $h(t) \in \mathcal{L}[0,1]$ *and* α, r *are two constants such that* $\alpha > 1 \geq r \geq 0$. *Then*

$$D^{r}_{0^+}\int_0^t (t-s)^{\alpha-1}h(s)ds = \frac{\Gamma\alpha}{\Gamma(\alpha-r)}\int_0^t (t-s)^{\alpha-r-1}h(s)ds.$$

Lemma 2.8 - *Let* $1 < \alpha \leq 2$, $0 < b_o < 1$, $\sigma = \dfrac{\Gamma\alpha}{\Gamma(\alpha-r)}$, $a_o = (1+\sigma)$ *and* $h \in C[0,1]$. *Then the unique solution of the BVP*

$$\left.\begin{aligned} D^{\alpha}u(t) + h(t) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u(1) + D^{r}u(1) = \int_0^1 p(s)u(s)ds, \end{aligned}\right\} \tag{2.9}$$

is given by

$$u(t) = \int_0^1 G_o(t,s)h(s)ds,$$

where

$$G_o(t,s) = \begin{cases} \dfrac{\sigma t^{\alpha-1}(1-s)^{\alpha-r-1} + t^{\alpha-1}(1-s)^{\alpha-1} - a_o(t-s)^{\alpha-1}}{a_o(1-b_o)\Gamma\alpha}, & s \leq t, \\ \dfrac{\sigma t^{\alpha-1}(1-s)^{\alpha-r-1} + t^{\alpha-1}(1-s)^{\alpha-1}}{a_o(1-b_o)\Gamma\alpha}, & t \leq s, \end{cases} \tag{2.10}$$

Proof: By Lemma (2.5), the BVP(2.9) can be reduced to an equivalent integral equation

$$u(t) \quad = \quad -I^{\alpha}h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$$

$$= \ -\frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}h(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}.$$

Using boundary condition $u(0) = 0$ with $\alpha \leq 2$, we have $c_2 = 0$ and

$$u(t) = -\frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}h(s)ds + c_1 t^{\alpha-1}.$$

$$u(1) = -\frac{1}{\Gamma\alpha}\int_0^1 (1-s)^{\alpha-1}h(s)ds + c_1. \tag{2.11}$$

$$D^r u(t) = -\frac{1}{\Gamma(\alpha-r)}\int_0^t (t-s)^{\alpha-r-1}h(s)ds + c_1\frac{\Gamma\alpha}{\Gamma(\alpha-r)}t^{\alpha-r-1}.$$

$$D^r u(1) = -\frac{1}{\Gamma(\alpha-r)}\int_0^1 (1-s)^{\alpha-r-1}h(s)ds + c_1\frac{\Gamma\alpha}{\Gamma(\alpha-r)}.$$

Using boundary condition $u(1) + D^r u(1) = \int_0^1 p(s)u(s)ds$ and setting $\frac{\Gamma\alpha}{\Gamma(\alpha-r)} = \sigma$, we have

$$c_1(1+\sigma) = \int_0^1 p(s)u(s)ds + \frac{1}{\Gamma(\alpha-r)}\int_0^1 (1-s)^{\alpha-r-1}h(s)ds + \frac{1}{\Gamma\alpha}\int_0^1 (1-s)^{\alpha-1}h(s)ds,$$

$$\Longrightarrow c_1 = \frac{1}{a_o}\int_0^1 p(s)u(s)ds + \frac{1}{a_o\Gamma\alpha}\int_0^1 (1-s)^{\alpha-1}h(s)ds + \frac{1}{a_o\Gamma(\alpha-r)}\int_0^1 (1-s)^{\alpha-r-1}h(s)ds. \tag{2.12}$$

where $a_o = (1+\sigma)$.
Putting (2.12) into (2.11) gives

$$u(t) = -\frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}h(s)ds + \frac{t^{\alpha-1}}{a_o}\int_0^1 p(s)u(s)ds + \frac{1}{a_o\Gamma\alpha}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}h(s)ds + \frac{1}{a_o\Gamma(\alpha-r)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-r-1}h(s)ds. \tag{2.13}$$

Multiply both sides of (2.13) by $p(t)$ and then integrate with respect to t from 0 to 1, we have

$$\int_0^1 p(t)u(t)dt = -\frac{1}{\Gamma\alpha}\int_0^1 p(t)\int_0^t (t-s)^{\alpha-1}h(s)dsdt$$

$$+\frac{1}{a_o}\int_0^1 p(t)t^{\alpha-1}dt\cdot\int_0^1 p(s)u(s)ds$$

$$+\frac{1}{a_o}\int_0^1 p(t)t^{\alpha-1}dt\cdot\frac{1}{\Gamma\alpha}\int_0^1 (1-s)^{\alpha-1}h(s)ds$$

$$+\frac{1}{a_o\Gamma(\alpha-r)}\int_0^1 p(t)t^{\alpha-1}dt\cdot\int_0^1 (1-s)^{\alpha-r-1}h(s)ds.$$

If we set $\int_0^1 p(t)u(t)dt = \delta_1$ and $\frac{1}{a_o}\int_0^1 p(t)t^{\alpha-1}dt = b_o$, then we have

$$\delta_1(1-b_o) = -\frac{1}{\Gamma\alpha}\int_0^1 p(t)\int_0^t (t-s)^{\alpha-1}h(s)dsdt$$

$$+\frac{b_o}{\Gamma\alpha}\int_0^1 (1-s)^{\alpha-1}h(s)ds$$

$$+\frac{b_o}{\Gamma(\alpha-r)}\int_0^1 (1-s)^{\alpha-r-1}h(s)ds.$$

$$\delta_1 = -\frac{1}{(1-b_o)\Gamma\alpha}\int_0^1 p(t)\int_0^t (t-s)^{\alpha-1}h(s)dsdt$$

$$+\frac{b_o}{(1-b_o)\Gamma\alpha}\int_0^1 (1-s)^{\alpha-1}h(s)ds$$

$$+\frac{b_o}{(1-b_o)\Gamma(\alpha-r)}\int_0^1 (1-s)^{\alpha-r-1}h(s)ds. \qquad (2.14)$$

Putting (2.14) into (2.13) gives

$$u(t) = -\frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}h(s)ds$$

$$+\frac{b_o}{a_o(1-b_o)\Gamma(\alpha-r)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-r-1}h(s)ds$$

$$+\frac{b_o}{a_o(1-b_o)\Gamma\alpha}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}h(s)ds$$

$$- \frac{b_o}{(1-b_o)\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}h(s)ds + \frac{1}{a_o\Gamma\alpha}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}h(s)ds$$

$$+ \frac{1}{a_o\Gamma(\alpha-r)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-r-1}h(s)ds.$$

$$= \frac{-1}{(1-b_o)\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}h(s)ds + \frac{1}{a_o(1-b_o)\Gamma\alpha}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}h(s)ds$$

$$+ \frac{1}{a_o(1-b_o)\Gamma(\alpha-r)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-r-1}h(s)ds.$$

$$= \frac{-1}{a_o(1-b_o)\Gamma\alpha}\int_0^t a_o(t-s)^{\alpha-1}h(s)ds + \frac{1}{a_o(1-b_o)\Gamma\alpha}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}h(s)ds$$

$$+ \frac{\sigma}{a_o(1-b_o)\Gamma\alpha}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-r-1}h(s)ds$$

$$= \frac{1}{a_o(1-b_o)\Gamma\alpha}\int_0^t \left[\sigma t^{\alpha-1}(1-s)^{\alpha-r-1} + t^{\alpha-1}(1-s)^{\alpha-1} - a_o(t-s)^{\alpha-1}\right] h(s)ds$$

$$+ \frac{1}{a_o(1-b_o)\Gamma\alpha}\int_t^1 \left[\sigma t^{\alpha-1}(1-s)^{\alpha-r-1} + t^{\alpha-1}(1-s)^{\alpha-1}\right] h(s)ds \qquad (2.15)$$

$$= \int_0^1 G_o(t,s)h(s)ds.$$

This completes the proof. □

Lemma 2.16 - *The function $G_o(t,s)$ defined by (2.10) is continuous and satisfies the following properties:*

(i) $G_o(t,s) \geq 0 \;\; \forall\, t,s \in [0,1]$ *and* $G_o(t,s) > 0 \;\; \forall\, t,s \in (0,1)$.

(ii) $G_o(t,s) \leq G_o(s,s) = \dfrac{\sigma s^{\alpha-1}(1-s)^{\alpha-r-1} + s^{\alpha-1}(1-s)^{\alpha-1}}{a_o(1-b_o)\Gamma\alpha}$, *for all* $t,s \in [0,1]$.

(iii) $\max\limits_{0\leq t\leq 1}\displaystyle\int_0^1 G_o(t,s)ds = \frac{\alpha\sigma + (\alpha-r) - a_o(\alpha-r)}{\alpha(\alpha-r)a_o(1-b_o)\Gamma\alpha}$.

(iv) There exists a positive function $\eta(s)$ such that

$$\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} G_o(t,s) \geq \eta(s) \max_{0\leq t\leq 1} G_o(t,s) = \eta(s)G_o(s,s),$$

for $\frac{1}{4}\leq t\leq\frac{3}{4}$, $s\in(0,1)$ *and* $0<\eta(s)<1$, *where*

$$\eta(s)=\begin{cases}\dfrac{(\frac{3}{4})^{\alpha-1}[\sigma(1-s)^{\alpha-r-1}+(1-s)^{\alpha-1}]-a_o(\frac{3}{4}-s)^{\alpha-1}}{s^{\alpha-1}[\sigma(1-s)^{\alpha-r-1}+(1-s)^{\alpha-1}]}, & s\in[\dfrac{1}{4},\dfrac{3}{4}],\\[2ex] \dfrac{1}{(4s)^{\alpha-1}}, & s\in[\dfrac{3}{4},1).\end{cases}$$

Proof: - (i) *It is standard and omitted.*

(ii) For $t\leq s$, we have

$$\begin{aligned}\frac{G_o(t,s)}{G_o(s,s)} &= \frac{\sigma t^{\alpha-1}(1-s)^{\alpha-r-1}+t^{\alpha-1}(1-s)^{\alpha-1}}{\sigma s^{\alpha-1}(1-s)^{\alpha-r-1}+s^{\alpha-1}(1-s)^{\alpha-1}},\\ &= \frac{t^{\alpha-1}[\sigma(1-s)^{\alpha-r-1}+(1-s)^{\alpha-1}]}{s^{\alpha-1}[\sigma(1-s)^{\alpha-r-1}+(1-s)^{\alpha-1}]},\\ &= \frac{t^{\alpha-1}}{s^{\alpha-1}},\\ &\leq 1.\end{aligned}$$

Similarly, for $s\leq t$, we have $\dfrac{G_o(t,s)}{G_o(s,s)}\leq 1$.

Hence, $G_o(t,s)\leq G_o(s,s)=\dfrac{s^{\alpha-1}[\sigma(1-s)^{\alpha-r-1}+(1-s)^{\alpha-1}]}{a_o(1-b_o)\Gamma\alpha}$,

for all $t,s\in[0,1]$.

(iii) Let

$$g_1(t,s)=\frac{\sigma t^{\alpha-1}(1-s)^{\alpha-r-1}+t^{\alpha-1}(1-s)^{\alpha-1}-a_o(t-s)^{\alpha-1}}{a_o(1-b_o)\Gamma\alpha},\quad 0\leq s\leq t,$$

$$g_2(t,s)=\frac{\sigma t^{\alpha-1}(1-s)^{\alpha-r-1}+t^{\alpha-1}(1-s)^{\alpha-1}}{a_o(1-b_o)\Gamma\alpha},\quad t\leq s\leq 1.$$

$$\int_0^t g_1(t,s)ds =$$

$$\frac{1}{a_o(1-b_o)\Gamma\alpha}\left[-\frac{\sigma t^{\alpha-1}(1-t)^{\alpha-r}}{(\alpha-r)}-\frac{t^{\alpha-1}(1-t)^{\alpha}}{\alpha}+\frac{\sigma t^{\alpha-1}}{(\alpha-r)}+\frac{t^{\alpha-1}}{\alpha}-\frac{a_o t^{\alpha}}{\alpha}\right].$$

$$\int_t^1 g_2(t,s)ds=\frac{1}{a_o(1-b_o)\Gamma\alpha}\left[\frac{\sigma t^{\alpha-1}(1-t)^{\alpha-r}}{(\alpha-r)}+\frac{t^{\alpha-1}(1-t)^{\alpha}}{\alpha}\right].$$

$$\int_0^1 G_o(t,s)ds = \frac{1}{a_o(1-b_o)\Gamma\alpha}\left[\frac{\sigma t^{\alpha-1}}{(\alpha-r)} + \frac{t^{\alpha-1}}{\alpha} - \frac{a_o t^{\alpha}}{\alpha}\right].$$
$$= \frac{\alpha\sigma t^{\alpha-1} + (\alpha-r)t^{\alpha-1} - a_o(\alpha-r)t^{\alpha}}{\alpha(\alpha-r)a_o(1-b_o)\Gamma\alpha}.$$
$$\Longrightarrow \max_{0\le t\le 1}\int_0^1 G_o(t,s)ds = \frac{\alpha\sigma + (\alpha-r) - a_o(\alpha-r)}{\alpha(\alpha-r)a_o(1-b_o)\Gamma\alpha}.$$

(iv) *The proof is similar to that of Lemma(2.4) in [5] and so we omit details.* This completes the proof. □

By Lemma(2.8), the solution $u(t)$ of the BVP(1.1) has an integral representation

$$u(t) = \lambda\int_0^1 G_o(t,s)w(s)f(s,u(s))ds. \tag{2.17}$$

Let $\mathcal{B}^* = C[0,1]$ be a Banach space equipped with the norm $\|u\| = \max_{0\le t\le 1}|u(t)|$ and $\mathcal{K}_o \subset \mathcal{B}^*$ be a cone defined by

$$\mathcal{K}_o = \left\{u \in \mathcal{B}^* : u(t) \ge 0 \text{ and } \min_{\frac{1}{4}\le t\le\frac{3}{4}} u(t) \ge \eta(s)\|u\|\right\}.$$

Define an integral operator $\mathcal{A} : \mathcal{K}_o \longrightarrow \mathcal{B}^*$ by

$$\mathcal{A}u(t) = \lambda\int_0^1 G_o(t,s)w(s)f(s,u(s))ds, \quad u \in \mathcal{K}_o. \tag{2.18}$$

Clearly, the solutions of the BVP(1.1) are the fixed points of the operator equation

$$u = \mathcal{A}u.$$

Lemma 2.19 *(see [2])- Let the operator $\mathcal{A}$ be defined as in (2.18). Then $\mathcal{A} : \mathcal{K}_o \longrightarrow \mathcal{K}_o$ is completely continuous.*

We state the Krasnosel'skii fixed-point theorem.

Theorem 2.20 *(see [17]) - Let $\mathcal{B}^*$ be a Banach Space and $\mathcal{K}_o \subset \mathcal{B}^*$ be a cone in $\mathcal{B}^*$. Assume Ω_1, Ω_2 are open subsets of $\mathcal{B}^*$ such that $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. If $\mathcal{A} : \mathcal{K}_o \cap (\overline{\Omega}_2\backslash\Omega_1) \longrightarrow \mathcal{K}_o$ is a completely continuous operator such that either*

(i) $\|\mathcal{A}u\| \le \|u\|$, $u \in \mathcal{K}_o \cap \partial\Omega_1$ and $\|\mathcal{A}u\| \ge \|u\|$, $u \in \mathcal{K}_o \cap \partial\Omega_2$, or

(ii) $\|\mathcal{A}u\| \ge \|u\|$, $u \in \mathcal{K}_o \cap \partial\Omega_1$ and $\|\mathcal{A}u\| \le \|u\|$, $u \in \mathcal{K}_o \cap \partial\Omega_2$ holds,

then $\mathcal{A}$ has a fixed point in $\mathcal{K}_o \cap (\overline{\Omega}_2\backslash\Omega_1)$.

3. Existence Results

Theorem 3.1 - *Assume conditions $C_1 - C_5$ are satisfied and let*

$$\left(\eta(s)\int_{1/4}^{3/4} G_o(s,s)w(s)ds\right)\mathcal{M}_o > \left(\int_0^1 G_o(s,s)w(s)ds\right)\mathcal{L}_o.$$

Then the BVP(1.1) has at least one positive solution provided

$$\frac{1}{\left(\eta(s)\int_{1/4}^{3/4} G_o(s,s)w(s)ds\right)\mathcal{M}_o} < \lambda < \frac{1}{\left(\int_0^1 G_o(s,s)w(s)ds\right)\mathcal{L}_o}. \tag{3.2}$$

Proof: Let λ be given as in (3.2) and choose $\varepsilon > 0$ such that

$$\frac{1}{\left(\eta(s)\int_{1/4}^{3/4} G_o(s,s)w(s)ds\right)(\mathcal{M}_o - \varepsilon)} \leq \lambda \leq \frac{1}{\left(\int_0^1 G_o(s,s)w(s)ds\right)(\mathcal{L}_o + \varepsilon)}. \tag{3.3}$$

By assumption C_4, there exists a constant $\delta_o > 0$ such that $f(t,u) \leq (\mathcal{L}_0 + \varepsilon)u$, for $0 < u \leq \delta_o$. Let $u \in \mathcal{K}_o$ such that $\|u\| = \delta_o$. Then we have

$$\begin{aligned}
\mathcal{A}u(t) &= \lambda\int_0^1 G_o(t,s)w(s)f(s,u(s))ds. \\
\|\mathcal{A}u\| &\leq \lambda\int_0^1 G_o(s,s)w(s)f(s,u(s))ds, \\
&\leq \lambda\int_0^1 G_o(s,s)w(s)(\mathcal{L}_0+\varepsilon)uds, \\
&\leq \delta_o = \|u\|. \\
\implies \|\mathcal{A}u\| &\leq \|u\|.
\end{aligned}$$

Setting $\Omega_1 = \{u \in \mathcal{B}^* : \|u\| < \delta_o\}$, then $\|\mathcal{A}u\| \leq \|u\|$, for $u \in (\mathcal{K}_o \cap \partial\Omega_1)$. By assumption C_5, there exists a constant $\kappa_1 > 0$ such that

$f(t,u) \geq (\mathcal{M}_o - \varepsilon)u$, for all $u \geq \kappa_1$.
Let $\delta_2 = \max\left\{2\delta_o \ , \ \dfrac{\kappa_1}{\eta(s)}\right\}$. Then for $u \in \mathcal{K}_o$ with $\|u\| = \delta_2$, we have

$$\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} u(t) \geq \eta(s)\|u\| \geq \kappa_1 \ \text{ and}$$

$$\begin{aligned}
\mathcal{A}u(t) &= \lambda\int_0^1 G_o(t,s)w(s)f(s,u(s))ds, \\
&\geq \lambda\int_0^1 G_o(t,s)w(s)(\mathcal{M}_o - \varepsilon)uds, \\
&\geq \lambda\eta(s)\int_{\frac{1}{4}}^{\frac{3}{4}} G_o(s,s)w(s)(\mathcal{M}_o - \varepsilon)uds, \\
&\geq \lambda\eta(s)\int_{\frac{1}{4}}^{\frac{3}{4}} G_o(s,s)w(s)(\mathcal{M}_o - \varepsilon)\|u\|ds \\
&\geq \|u\|.
\end{aligned}$$

$$\Longrightarrow \|\mathcal{A}u\| \geq \|u\|.$$

Setting $\Omega_2 = \{u \in \mathcal{B}^* : \|u\| < \delta_2\}$, then $\|\mathcal{A}u\| \geq \|u\|$, for $u \in (\mathcal{K}_o \cap \partial\Omega_2)$.
By the application of part (i) of Theorem(2.20), we conclude that operator $\mathcal{A}$ has a fixed point in $\mathcal{K}_o \cap (\overline{\Omega}_2 \backslash \Omega_1)$. □

Theorem 3.4 - *Assume conditions $C_1 - C_5$ are satisfied and let*

$$\left(\eta(s)\int_{1/4}^{3/4} G_o(s,s)w(s)ds\right)\mathcal{L}_o > \left(\int_0^1 G_o(s,s)w(s)ds\right)\mathcal{M}_o.$$

Then the BVP(1.1) has at least one positive solution provided

$$\frac{1}{\left(\eta(s)\int_{1/4}^{3/4} G_o(s,s)w(s)ds\right)\mathcal{L}_o} < \lambda < \frac{1}{\left(\int_0^1 G_o(s,s)w(s)ds\right)\mathcal{M}_o}. \tag{3.5}$$

Proof: Let λ be given as in (3.5) and choose $\varepsilon > 0$ such that

$$\frac{1}{\left(\eta(s)\int_{1/4}^{3/4} G_o(s,s)w(s)ds\right)(\mathcal{L}_o - \varepsilon)} \leq \lambda \leq \frac{1}{\left(\int_0^1 G_o(s,s)w(s)ds\right)(\mathcal{M}_o + \varepsilon)}. \tag{3.6}$$

By assumption C_4, there exists a constant $\delta_o > 0$ such that
$f(t,u) \geq (\mathcal{L}_o - \varepsilon)u$, for $0 < u \leq \delta_o$.
Let $u \in \mathcal{K}_o$ such that $\|u\| = \delta_o$. Then we have $\min_{\frac{1}{4}\leq t\leq \frac{3}{4}} u(t) \geq \eta(s)\|u\|$ and

$$\begin{aligned}
\mathcal{A}u(t) &= \lambda \int_0^1 G_o(t,s)w(s)f(s,u(s))ds, \\
&\geq \lambda \int_0^1 G_o(t,s)w(s)(\mathcal{L}_0 - \varepsilon)uds, \\
&\geq \lambda\eta(s)\int_{\frac{1}{4}}^{\frac{3}{4}} G_o(s,s)w(s)ds \cdot \|u\|, \\
&\geq \|u\|.
\end{aligned}$$

$$\implies \|\mathcal{A}u\| \geq \|u\|.$$

Setting $\Omega_1 = \{u \in \mathcal{B}^* : \|u\| < \delta_o\}$, then $\|\mathcal{A}u\| \geq \|u\|$, for $u \in (\mathcal{K}_o \cap \partial\Omega_1)$.
By assumption C_5, there exists a constant $\kappa_1 > 0$ such that
$f(t,u) \leq (\mathcal{M}_o + \varepsilon)u$, for all $u \geq \kappa_1$.
We consider two possibilities:

Case 1: Suppose f is bounded. Then there exists a constant $\mathcal{N}^* > 0$ such that $f(t,u) \leq \mathcal{N}^*$, for $0 < u < \infty$.

Let $\delta_2 = \max\left\{2\delta_o,\ \lambda\mathcal{N}^*\int_0^1 G_o(s,s)w(s)ds\right\}$. Then for $u \in \mathcal{K}_o$ with $\|u\| = \delta_2$,
we have

$$\begin{aligned}
\mathcal{A}u(t) &= \lambda \int_0^1 G_o(t,s)w(s)f(s,u(s))ds. \\
\|\mathcal{A}u\| &\leq \lambda \int_0^1 G_o(s,s)w(s)f(s,u(s))ds. \\
&\leq \lambda \int_0^1 G_o(s,s)w(s) \cdot \mathcal{N}^* ds \\
&\leq \lambda\mathcal{N}^* \int_0^1 G_o(s,s)w(s)ds. \\
&\leq \delta_2 \\
&= \|u\|.
\end{aligned}$$

$$\Longrightarrow \|Au\| \;\leq\; \|u\|.$$

Setting $\Omega_2 = \{u \in \mathcal{B}^* : \|u\| < \delta_2\}$, then $\|\mathcal{A}u\| \leq \|u\|$, for $u \in (\mathcal{K}_o \cap \partial\Omega_2)$.

Case 2: Suppose f is not bounded and let $\delta_2 \geq \max\{2\delta_o, \;\; \kappa_1\}$ be chosen such that $\kappa_1 \leq u \leq \delta_2$. Then for $u \in \mathcal{K}_o$ with $\|u\| = \delta_2$, we have

$$\begin{aligned}
\mathcal{A}u(t) &= \lambda \int_0^1 G_o(t,s)w(s)f(s,u(s))ds. \\
\|\mathcal{A}u\| &\leq \lambda \int_0^1 G_o(s,s)w(s)f(s,u(s))ds, \\
&\leq \lambda \int_0^1 G_o(s,s)w(s)(\mathcal{M}_0 + \varepsilon)uds, \\
&\leq \lambda \int_0^1 G_o(s,s)w(s)(\mathcal{M}_0 + \varepsilon)ds \cdot \delta_2, \\
&\leq \lambda \int_0^1 G_o(s,s)w(s)(\mathcal{M}_0 + \varepsilon)\|u\|. \\
\Longrightarrow \|\mathcal{A}u\| &\leq \|u\|.
\end{aligned}$$

Setting $\Omega_2 = \{u \in \mathcal{B}^* : \|u\| < \delta_2\}$, then $\|\mathcal{A}u\| \leq \|u\|$, for $u \in (\mathcal{K}_o \cap \partial\Omega_2)$. Therefore, in any of the two possibilities, we have

$$\|\mathcal{A}u\| \leq \|u\|, \;\; \text{for} \;\; u \in (\mathcal{K}_o \cap \partial\Omega_2).$$

By the application of part (ii) of Theorem(2.20), the operator $\mathcal{A}$ has a fixed point in $\mathcal{K}_o \cap (\overline{\Omega}_2 \backslash \Omega_1)$. □

4. Example

Consider the nonlinear eigenvalue problem:

$$\left.\begin{aligned}
D^{\frac{3}{2}}u(t) + \lambda\frac{(1-t)}{2}\left(10 + 5t - 20t^2e^{-u}\right)u = 0, \;\; t \in (0,1), \\
u(0) = 0, \quad u(1) + D^{\frac{1}{2}}u(1) = \int_0^1 t^{\frac{1}{2}} \cdot u(t)dt.
\end{aligned}\right\} \tag{4.1}$$

Here, $\alpha = \frac{3}{2}$, $r = \frac{1}{2}$, $w(t) = \frac{(1-t)}{2}$, $p(t) = t^{\frac{1}{2}}$.
Now, condition C_2 holds for $w(t) = \frac{(1-t)}{2} \neq 0$, for all $t \in (0,1)$.
Let $u \in [0,\infty)$ and $t \in [0,1]$. Then $f(t,u) = u\left(10 + 5t - 20t^2 e^{-u}\right)$ is continuous and condition C_1 holds.
By simple computation, we have $\sigma = \frac{\Gamma\alpha}{\Gamma(\alpha - r)} = \frac{\sqrt{\pi}}{2} = 0.886226925$,

$a_o = (1+\sigma) = 1.886226925$, $\quad b_o = \frac{1}{a_o}\int_0^1 p(t) t^{\alpha-1} dt = 0.265079452$,

$\mathcal{L}_o = \lim_{u\to 0^+} \frac{f(t,u)}{u} = 2.50$, $\quad \left(\int_0^1 G_o(s,s) w(s) ds\right)\mathcal{L}_o = 0.440244837$,

$\mathcal{M}_o = \lim_{u\to\infty} \frac{f(t,u)}{u} = 13.750$, $\quad \left(\eta(s)\int_{1/4}^{3/4} G_o(s,s) w(s) ds\right)\mathcal{M}_o = 0.589140608$,

$$\left[\left(\int_0^1 G_o(s,s) w(s) ds\right)\mathcal{L}_o\right]^{-1} = 2.271463322 \approx 2.2715.$$

$$\left[\left(\eta(s)\int_{1/4}^{3/4} G_o(s,s) w(s) ds\right)\mathcal{M}_o\right]^{-1} = 1.697387663 \approx 1.6974.$$

By Theorem(3.1), we have $0.589140608 > 0.440244837$ and $1.6974 < \lambda < 2.2715$. Hence, the BVP(4.1) has at least one positive solution for each $\lambda \in (1.6974,\ 2.2715)$.

Acknowledgments

The authors are very grateful to the anonymous referees for their valuable comments and suggestions leading to the improvement of the original manuscript.

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