Transactions of the Nigerian Association of Mathematical Physics
*Volume 16, (July–Sept. 2021 Issue), pp*129 – 140
©*Trans. of NAMP*

On Blow Up of Positive Initial Energy Solution of a Nonlinear Wave Equation with Nonlinear Source and Boundary Damping Terms

Paul A. Ogbiyele[1] and Peter O. Arawomo[2]

[1,2]Department of Mathematics, University of Ibadan, Nigeria
[1]paulogbiyele@yahoo.com, [2]po.arawomo@ui.edu.ng

Abstract

In this paper, we consider a nonlinear wave equation having nonlinear source and boundary damping terms

$$u_{tt} - \operatorname{div}\Big[|\nabla u|^{\gamma}\nabla u + (1+|\nabla u_t|^r)\nabla u_t\Big] = g(x,u) \quad in \quad (0,\infty)\times\Omega$$
$$u = 0 \qquad on \quad [0,\infty]\times\Gamma_0$$
$$|\nabla u|^{\gamma}\frac{\partial u}{\partial \nu} + (1+|\nabla u_t|^r)\frac{\partial u_t}{\partial \nu} + f(x,u_t) = 0 \quad on \quad [0,\infty]\times\Gamma_1$$
$$u(x,0) = u_0, \qquad u_t(x,0) = u_1, \quad on \quad \Omega$$

and obtain blow up results under certain polynomial growth conditions on γ, r, m and p, where the polynomial growth order of the nonlinear functions g and f are $p+1$ and $m+1$ respectively. We obtain the blow up result using the perturbed energy technique.

Keywords: Non-linear boundary damping, Non-linear source, Positive initial energy, Potential well, Blow up.

Mathematics Subject Classification (2010):35B44, 35L72

1 Introduction

We are concerned with blow up of solutions to nonlinear wave equations of the form

$$\begin{cases} u_{tt} - \operatorname{div}\Big[|\nabla u|^{\gamma}\nabla u + (1+|\nabla u_t|^r)\nabla u_t\Big] = g(x,u) & in \quad (0,\infty)\times\Omega \\ u = 0 & on \quad [0,\infty]\times\Gamma_0 \\ |\nabla u|^{\gamma}\dfrac{\partial u}{\partial \nu} + (1+|\nabla u_t|^r)\dfrac{\partial u_t}{\partial \nu} + f(x,u_t) = 0 & on \quad [0,\infty]\times\Gamma_1 \\ u(x,0) = u_0, \qquad u_t(x,0) = u_1, & on \quad \Omega, \end{cases} \tag{1.1}$$

Correspondence Author: P.A. Ogbiyele, Email: paulogbiyele@yahoo.com,

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma$ such that $\Gamma = \Gamma_0 \cup \Gamma_1$ and satisfying $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\lambda_{n-1}(\Gamma_0) > 0$ (where λ_{n-1} denotes the (n-1) dimensional Lebesgue measure on $\partial\Omega$). The derivative $\frac{\partial}{\partial\nu}$ is the unit outward normal derivative to Γ.

Equations of the form (1.1) arise in the study of nonlinear wave equations describing the motion of a viscoelastic solid made up of materials of the rate type. There is an extensive literature on blow up of solutions of non-linear wave equations having negative initial energy and of the form

$$\begin{cases} u_{tt} - \Delta u_t - \operatorname{div}\Big[|\nabla u|^\gamma \nabla u + |\nabla u_t|^r \nabla u_t\Big] + |u_t|^m u_t = |u|^p u \quad x \in \Omega, \quad t > 0 \\ u(x,t)|_{\partial\Omega} = 0, \quad t > 0 \qquad u(x,0) = u_0, \qquad u_t(x,0) = u_1, \qquad x \in \Omega. \end{cases} \tag{1.2}$$

Georgiev and Todorova [3] considered global existence and blow up of solutions to (1.2) for $\gamma = 0$, $m > 0$ and in the absence of the strong damping terms. In considering the relationship between m and p, they showed that for $m \geq p$ with negative initial energy, the solution is global in time and for $p > m$ the solution cannot be global when the initial energy is sufficiently negative. Thus extending the result of Levin [5, 6], where $m = 0$.

In [18], Yang obtained blow up of solutions to (1.2) under the condition $p > \max\{\gamma, m\}$ and where the blow up time depends on $|\Omega|$.

Messaoudi and Said-Houari [10] studied a class of nonlinear wave equations having the form (1.2) but in the absence of the strong damping term and obtained blow up result for $p > \max\{\gamma, m\}$ where the blow up result holds regardless of the size of Ω. Thus extending the result of Yang [18].

Liu and Wang [8] considered a class of wave equations of the form (1.2) and established blow up results for certain solutions with non-positive initial energy as well as positive initial energy. This further improves the results of Yang [18] and Messaoudi and Said-Houari [10].

In [14], Piskin investigated the energy decay of solutions for quasi-linear hyperbolic equations of the form (1.2) with nonlinear damping and source terms and obtained blow up result for the case $m = 0$, using the concavity method. Jeong, et al. [4] considered global nonexistence of solutions to a quasi-linear wave equation of the form (1.2) with acoustic boundary conditions and satisfying $p > \max\{\gamma, m\}$ and $\gamma > r$.

The author in [12], considered global existence and blow up of positive initial energy solution of a quasilinear wave equation of the form (1.2) with initial boundary conditions and nonlinear damping and source terms. The result includes a more general case of nonlinear wave equations which exhibit space dependent γ-Laplacian operator and where the nonlinear damping and source terms have varying coefficients. He obtained blow up result under polynomial growth conditions satisfying $p > \max\{\gamma, m\}$ and $\gamma > r$. For other related results, see[9, 13, 16, ?] and for a review on recent results regarding global existence, blow up and energy decay of solutions to wave equations in bounded domains see [11].

In this paper, we obtain blow up of positive initial energy solution to the boundary value problem (1.1), using the perturbed energy method.

2 Preliminaries

In this section, we state some basic assumptions used in this paper. For simplicity, we introduce the following notations.

$L^p(\Omega)$, $1 \leq p \leq \infty$, the Lebesgue space with norm $\|\cdot\|_p$ and $L^p(\Gamma_1, 1)$, the standard L^p space associated to λ_{n-1}, that is $L^p(\Gamma_1) = L^p(\Gamma_1, 1)$ and $\|\cdot\|_{p,\Gamma_1} = \|\cdot\|_{L^p(\Gamma_1)}$. We also denote by $W^{k,p}(\Omega)$ the Banach space of functions in $L^p(\Omega)$ with $k (k \in \mathbb{N})$ generalized derivatives and consider the Banach space

$$W^{1,\gamma+2}_{\Gamma_0}(\Omega) = \{u \in W^{1,\gamma+2}(\Omega) : u|_{\Gamma_0} = 0\} \tag{2.1}$$

where $u|_{\Gamma_0}$ is given in the trace sense, and $W^{1,\gamma+2}_{\Gamma_0}(\Omega)$ is the closure of $C^1_0(\Omega\cup\Gamma_0)$ with respect to the norm of $W^{1,\gamma+2}(\Omega)$. Considering the fact that $\lambda_{n-1}(\Gamma_0)$ is strictly positive, the Poincaré inequality can be applied to the space $W^{1,\gamma+2}_{\Gamma_0}(\Omega)$.

Let K be the smallest positive constant such that

$$\|u\|_{p+2} \le K\|\nabla u\|_{\gamma+2}, \tag{2.2}$$

for all $u \in W^{1,\gamma+2}_{\Gamma_0}(\Omega)$, then we have the following embedding

$$W^{1,\gamma+2}_{\Gamma_0}(\Omega) \hookrightarrow L^{p+2}(\Omega) \tag{2.3}$$

where the constants p, γ satisfy $p \ge \gamma$.

We state the following assumptions on the nonlinear functions g and f representing the nonlinear source and boundary damping terms respectively.

(A_1) $g \in C(\mathbb{R})$, $g(\cdot,s)s \ge 0$, and there exist positive constants λ_1 and λ_2 such that

$$\lambda_1|s|^{p+1} \le |g(\cdot,s)| \le \lambda_2|s|^{p+1}, \quad s\in\mathbb{R} \tag{2.4}$$

where $0<p<+\infty$ if $n\le\gamma+2$ and $2<p+2\le\frac{n(\gamma+2)}{n-\gamma-2}$ when $n>\gamma+2$

(A_2) $f \in C(\mathbb{R})$, $f(\cdot,s)s \ge 0$, and there exist positive constants ρ_1 and ρ_2 such that

$$\rho_1|s|^{m+1} \le |f(\cdot,s)| \le \rho_2|s|^{m+1}, \quad s\in\mathbb{R} \tag{2.5}$$

where $0<m<+\infty$ if $n\le\gamma+2$ and $2<m+2\le\frac{(n-1)(\gamma+2)}{n-\gamma-2}$ when $n>\gamma+2$

We define the energy function associated to problem (1.1) by

$$E(t) := \frac{1}{2}\|u_t\|^2 + \frac{1}{\gamma+2}\|\nabla u\|^{\gamma+2}_{\gamma+2} - \int_\Omega\int_0^u g(\cdot,y)dydx \tag{2.6}$$

and for the energy function (2.6), we have the following result.

Lemma 2.1. *Assume that (A_1)-(A_2) hold. Let u be a solution of (1.1), then the energy function $E(t)$ of the problem (1.1) is defined by (2.6). In addition, $E(t)$ is non increasing and satisfies*

$$E'(t) = -\|\nabla u_t\|^2_2 - \|\nabla u_t\|^{r+2}_{r+2} - \int_{\Gamma_1} f(\cdot,u_t)\, u_t d\Gamma \tag{2.7}$$

Moreover, we have

$$E(t) \le E(0) \tag{2.8}$$

Proof. By multiplying (1.1) by u_t and integrating over Ω, we obtain the estimate (2.7) for any regular solution. Thus by using density arguments, we get the desired result. □

Now, we define

$$K_\infty := \sup_{0\ne u\in W^{1,\gamma+2}_{\Gamma_0}(\Omega)} \left(\frac{\frac{1}{p+2}\|u\|^{p+2}_{p+2}}{\|\nabla u\|^{p+2}_{\gamma+2}}\right). \tag{2.9}$$

Therefore, we have that

$$\frac{1}{p+2}\|u\|^{p+2}_{p+2} \le K_\infty\|\nabla u\|^{p+2}_{\gamma+2} \tag{2.10}$$

Also, consider the functional $J(u)$ defined by

$$J(u) := \frac{1}{\gamma+2}\|\nabla u\|^{\gamma+2}_{\gamma+2} - \frac{\lambda_2}{p+2}\|u\|^{p+2}_{p+2}, \qquad u\in W^{1,\gamma+2}_{\Gamma_0}(\Omega) \tag{2.11}$$

and re-express the energy associated to (1.1) as

$$E(t) = E(u,u_t) := \frac{1}{2}\|u_t\|^2 + J_p(u) \tag{2.12}$$

where the associated potential energy $J_p(u)$ is defined by

$$J_p(u) = \frac{1}{\gamma+2}\|\nabla u\|_{\gamma+2}^{\gamma+2} - \int_\Omega \int_0^u g(\cdot,y)dydx,$$

Then substituting (2.10) in (2.11), we have that (2.12) yields

$$E(t) \geq \frac{1}{\gamma+2}\|\nabla u\|_{\gamma+2}^{\gamma+2} - \lambda_2 K_\infty \|\nabla u\|_{\gamma+2}^{p+2}$$

Now, setting

$$\xi = \|\nabla u\|_{\gamma+2}, \tag{2.13}$$

we obtain

$$E(t) \geq \frac{1}{\gamma+2}\xi^{\gamma+2} - \lambda_2 K_\infty \xi^{p+2} := h(\xi) \tag{2.14}$$

From (2.14), we have that the first positive zero of the function $h'(\xi)$ (the absolute maximum point of $h(\xi)$) is given by

$$\xi_\infty = \left[\frac{1}{(p+2)\lambda_2 K_\infty}\right]^{\frac{1}{p-\gamma}} \tag{2.15}$$

It can be verified that for $0 < \xi < \xi_\infty$, the function $h(\xi)$ is increasing and it is decreasing for $\xi > \xi_\infty$. The maximum mountain pass level of $J(u)$ is given by

$$h(\xi_\infty) = \frac{(p-\gamma)}{(p+2)(\gamma+2)}\left(\frac{1}{(p+2)\lambda_2 K_\infty}\right)^{\frac{\gamma+2}{p-\gamma}} := E_\infty \tag{2.16}$$

Lemma 2.2. *The potential well depth E_∞ is defined by*

$$E_\infty := \inf_{0\neq u \in W_{\Gamma_0}^{1,\gamma+2}(\Omega)} \sup_{\xi>0} J(\xi u) > 0 \tag{2.17}$$

Proof:
For $u \in W_{\Gamma_0}^{1,\gamma+2}(\Omega)$, using the fact that $\frac{d}{d\xi}J(\xi u)|_{\xi=\xi^*} = 0$, where

$$\xi^* = \xi(u) = \left[\frac{\|\nabla u\|_{\gamma+2}^{\gamma+2}}{\lambda_2 \|u\|_{p+2}^{p+2}}\right]^{\frac{1}{p-\gamma}}$$

we obtain

$$\sup_{\xi>0} J(\xi u) = J(\xi^* u) = \left[\frac{(p-\gamma)}{(\gamma+2)(p+2)}\right]\left[\frac{1}{\lambda_2(p+2)}\right]^{\frac{\gamma+2}{p-\gamma}}\left[\frac{\|\nabla u\|_{\gamma+2}^{p+2}}{\frac{1}{p+2}\|u\|_{p+2}^{p+2}}\right]^{\frac{\gamma+2}{p-\gamma}}$$

thus, from (2.9) we have

$$\begin{aligned}
&\inf_{0\neq u\in W_{\Gamma_0}^{1,\gamma+2}(\Omega)} \sup_{\xi>0} J(\xi u)\\
&= \left[\frac{(p-\gamma)}{(\gamma+2)(p+2)}\right]\left[\frac{1}{\lambda_2(p+2)}\right]^{\frac{\gamma+2}{p-\gamma}}\left[\inf_{0\neq u\in W_{\Gamma_0}^{1,\gamma+2}(\Omega)}\left(\frac{\|\nabla u\|_{\gamma+2}^{p+2}}{\frac{1}{p+2}\|u\|_{p+2}^{p+2}}\right)\right]^{\frac{\gamma+2}{p-\gamma}}\\
&= \left[\frac{(p-\gamma)}{(\gamma+2)(p+2)}\right]\left[\frac{1}{\lambda_2 K_\infty(p+2)}\right]^{\frac{\gamma+2}{p-\gamma}} = E_\infty
\end{aligned}$$

Remark 2.1. *It can be shown that E_∞ as defined in (2.16) is the mountain pass level associated to the elliptic problem*

$$
\begin{aligned}
-\Delta_\gamma u &= \lambda_2 |u|^p u \quad \text{in} \quad [0,\infty]\times\Omega \\
u &= 0, \quad \text{on} \quad [0,\infty]\times\Gamma_0, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on} \quad [0,\infty]\times\Gamma_1, \qquad \Delta_\gamma u = div(|\nabla u|^\gamma \nabla u)
\end{aligned}
$$

see [2, ?]. In this case E_∞ is equal to the number $\inf_{\zeta\in\Lambda}\sup_{t\in[0,1]} J(\zeta(t))$ where

$$\Lambda = \left\{\zeta \in C\big([0,1]; W^{1,\gamma+2}_{\Gamma_0}(\Omega)\big) : \zeta(0)=0,\ J(\zeta(1))<0\right\}$$

2.1 Local existence

Theorem 2.1. *Suppose that the assumptions (A_1)-(A_2) hold. If in addition the following conditions;*

$$\|\nabla u_0\|_{\gamma+2} < \xi_\infty, \quad E(0) < E_\infty$$

are satisfied on the initial data, then there exist a unique solution u of (1.1) for any $T>0$ such that

$$
\begin{aligned}
&u \in L^\infty([0,T]; W^{1,\gamma+2}_{\Gamma_0}(\Omega)) \cap L^\infty([0,T]; L^{p+2}(\Omega)) \\
&u_t \in L^\infty([0,T]; L^2(\Omega)) \cap L^{r+2}([0,T]; W^{1,r+2}_{\Gamma_0}(\Omega)) \cap L^{m+2}([0,T]\times\Gamma_1)
\end{aligned}
$$

For similar proofs see [12, 15, 17], hence we omit the proof here

3 Blow-up result

In this section, we shall discuss the blow up property of the solution to (1.1) having positive initial energy. To achieve this, we employ the idea of Georgiev and Todorova [3].

Lemma 3.1. *Assume that (A_1)-(A_2) hold. Let u be a solution of (1.1) with initial data satisfying*

$$E(0) < E_\infty \quad \text{and} \quad \|\nabla u_0\|_{\gamma+2} > \xi_\infty \qquad \forall t \in [0,T) \tag{3.1}$$

Then there exists a constant $\xi_1 > \xi_\infty$ such that

$$\|\nabla u\|_{\gamma+2} > \xi_1 \quad \text{for all} \quad t\in[0,T) \tag{3.2}$$

and moreover, the following inequality holds

$$\|u\|^{p+2}_{p+2} \geq (p+2)K_\infty \xi_1^{p+2} \tag{3.3}$$

We omit the proof here to avoid repetition of ideas, see [12, 17] for the proof .

Now, define the function $H(t)$ by

$$H(t) := E_\infty - E(t) \tag{3.4}$$

then, from (2.6), (2.8), (2.10) and (3.2), we have

$$
\begin{aligned}
0 < H(0) \leq H(t) &\leq E_\infty - \frac{1}{\gamma+2}\|\nabla u\|^{\gamma+2}_{\gamma+2} + \int_\Omega\int_0^u g(\cdot,y)dydx \\
&\leq -\left[\frac{\gamma+2}{p-\gamma}\right] E_\infty + \int_\Omega\int_0^u g(\cdot,y)dydx \\
&\leq \frac{\lambda_2}{p+2}\|u\|^{p+2}_{p+2} \leq \lambda_2 K_\infty \|\nabla u\|^{p+2}_{\gamma+2}
\end{aligned} \tag{3.5}
$$

Moreover, from (2.7) and assumption (A_2), the derivative $H'(t)$ satisfy

$$H'(t) \geq \|\nabla u_t\|^2 + \|\nabla u_t\|_{r+2}^{r+2} + \rho_1 \|u_t\|_{m+2,\Gamma_1}^{m+2} \tag{3.6}$$

Furthermore, define the function $L(t)$ by

$$L(t) := H^{1-\rho}(t) + \mu \int_\Omega u u_t dx \tag{3.7}$$

where ρ is a positive constant to be determined later.

Then, we have the following

Theorem 3.1. *Let $u(x,t)$ be a solution of the problem (1.1), assume that the conditions of Lemma 3.1 are satisfied. In addition, suppose that $g(u)$ satisfies*

(B_1) $\int_\Omega ug(\cdot,u)dx - q\int_\Omega \int_0^u g(\cdot,y)dydx \geq \eta_0 \|u\|_{p+2}^{p+2}$

for positive constants $q \in (\gamma+2, p+2)$ and $\eta_0 \geq \frac{\lambda_2(p+2-q)}{p+2}$. Then for $0 < m < \frac{p(\gamma+2)^2 + (p-\gamma)[\gamma(n-1)-2]}{n(p-\gamma)+(\gamma+2)^2}$ and $0 \leq r < \gamma$, there is a $T_{max} > 0$ such that the solution $u(x,t)$ blows up in finite time.

Proof of Theorem 3.1

Let u be a solution of (1.1), define the function

$$a(t) = \frac{1}{2}\|u(t)\|^2$$

where $t \in [0,T]$. In the presence of the nonlinear boundary damping term, we follow the idea of Todorova and Georgiev [3] and define the function as in (3.7) by

$$L(t) := H^{1-\rho}(t) + \mu a'(t) \tag{3.8}$$

where $\mu a'(t)$ is a small perturbation of the function $H^{1-\rho}(t)$ and μ is a small positive constant to be determined later. Hence, (3.8) yields,

$$L(t) = H^{1-\rho}(t) + \mu\left[\int_\Omega u u_t \, dx\right] \tag{3.9}$$

for suitable choice of ρ satisfying

$$0 < \rho \leq \min\left\{\frac{\gamma}{2(p+2)}, \ \frac{\gamma+2-s(m+2)}{(m+2)(\gamma+2)} - \left(\frac{1-s}{p+2}\right), \ \frac{\gamma-r}{(p+2)(r+2)}\right\} \tag{3.10}$$

and $s < 1$, both to be determined later. Then differentiating (3.9), we obtain

$$L'(t) = (1-\rho)H^{-\rho}(t)H'(t) + \mu\int_\Omega u_t^2 \, dx + \mu\int_\Omega u u_{tt} dx \tag{3.11}$$

Furthermore, the use of (1.1) and (3.11) gives

$$\begin{aligned} L'(t) \geq &(1-\rho)H^{-\rho}(t)H'(t) + \mu\int_\Omega u_t^2 dx - \mu\int_\Omega |\nabla u|^{\gamma+2} dx + \mu\int_\Omega ug(\cdot,u)dx \\ &- \mu\int_\Omega \nabla u \nabla u_t dx - \mu\int_\Omega |\nabla u_t|^r \nabla u_t \nabla u dx - \mu\int_{\Gamma_1} f(\cdot,u_t)u d\Gamma \end{aligned} \tag{3.12}$$

From (3.4) and the energy identity (2.6), we have that

$$q\int_\Omega\int_0^u g(\cdot,y)dydx = \frac{q}{2}\|u_t\|^2 + \frac{q}{\gamma+2}\|\nabla u\|_{\gamma+2}^{\gamma+2} + qH(t) - qE_\infty \tag{3.13}$$

for $\gamma+2 < q < p+2$. Then, using assumption (B_1), we obtain

$$\int_\Omega ug(\cdot,u)dx \geq \frac{q}{2}\|u_t\|^2 + \frac{q}{\gamma+2}\|\nabla u\|_{\gamma+2}^{\gamma+2} + qH(t) - qE_\infty + \eta_0\|u\|_{p+2}^{p+2} \tag{3.14}$$

substituting (3.14) into (3.12), we get

$$\begin{aligned} L'(t) \geq & (1-\rho)H^{-\rho}(t)H'(t) + \mu\left[\tfrac{q+2}{2}\right]\|u_t\|^2 + \mu\left[\tfrac{q-(\gamma+2)}{\gamma+2}\right]\|\nabla u\|_{\gamma+2}^{\gamma+2} \\ & + \mu\eta_0\|u\|_{p+2}^{p+2} - \mu\int_\Omega \nabla u\nabla u_t dx - \mu\int_\Omega |\nabla u_t|^r\nabla u_t\nabla u dx \\ & - \mu\int_{\Gamma_1} f(\cdot,u_t)ud\Gamma + \mu qH(t) - \mu qE_\infty \end{aligned} \tag{3.15}$$

Now using (2.15), (2.16) and (3.2), we re-express the third term on the right hand side of (3.15) as

$$\begin{aligned} & \mu\left[\tfrac{q-(\gamma+2)}{\gamma+2}\right]\|\nabla u\|_{\gamma+2}^{\gamma+2} \\ & = \mu\left[\tfrac{q-(\gamma+2)}{\gamma+2}\right]\left[\tfrac{\xi_1^{\gamma+2}-\xi_\infty^{\gamma+2}}{\xi_1^{\gamma+2}}\right]\|\nabla u\|_{\gamma+2}^{\gamma+2} + \mu\left[\tfrac{q-(\gamma+2)}{\gamma+2}\right]\left[\tfrac{\xi_\infty^{\gamma+2}}{\xi_1^{\gamma+2}}\right]\|\nabla u\|_{\gamma+2}^{\gamma+2} \\ & \geq \mu a_1\|\nabla u\|_{\gamma+2}^{\gamma+2} + \mu\tfrac{q-(\gamma+2)}{\gamma+2}\xi_\infty^{\gamma+2} \geq \mu a_1\|\nabla u\|_{\gamma+2}^{\gamma+2} + \mu\tfrac{[q-(\gamma+2)][p+2]}{p-\gamma}E_\infty \end{aligned} \tag{3.16}$$

Likewise using (2.15), (2.16) and (3.3), we re-express the fourth term on the right hand side of (3.15) as

$$\begin{aligned} \mu\eta_0\|u\|_{p+2}^{p+2} &= \mu\eta_0\left[\tfrac{\xi_1^{p+2}-\xi_\infty^{p+2}}{\xi_1^{p+2}}\right]\|u\|_{p+2}^{p+2} + \mu\eta_0\left[\tfrac{\xi_\infty^{p+2}}{\xi_1^{p+2}}\right]\|u\|_{p+2}^{p+2} \\ & \geq \mu\eta_1\|u\|_{p+2}^{p+2} + \mu\eta_0 K_\infty(p+2)\xi_\infty^{p+2} \\ & \geq \mu\eta_1\|u\|_{p+2}^{p+2} + \tfrac{\mu\eta_0(p+2)(\gamma+2)}{\lambda_2(p-\gamma)}E_\infty \end{aligned} \tag{3.17}$$

where we set $a_1 = \left[\frac{q-(\gamma+2)}{\gamma+2}\right]\left[\frac{\xi_1^{\gamma+2}-\xi_\infty^{\gamma+2}}{\xi_1^{\gamma+2}}\right] > 0$ and $\eta_1 = \eta_0\left[\frac{\xi_1^{p+2}-\xi_\infty^{p+2}}{\xi_1^{p+2}}\right] > 0$. Then using the estimate (3.16) and (3.17) in (3.15), we obtain

$$\begin{aligned} L'(t) \geq & (1-\rho)H^{-\rho}(t)H'(t) + \mu\left[\tfrac{q+2}{2}\right]\|u_t\|^2 + \mu a_1\|\nabla u\|_{\gamma+2}^{\gamma+2} + \mu\eta_1\|u\|_{p+2}^{p+2} \\ & + \mu\tfrac{[q-(\gamma+2)][p+2]}{p-\gamma}E_\infty + \tfrac{\mu\eta_0(p+2)(\gamma+2)}{\lambda_2(p-\gamma)}E_\infty - \mu qE_\infty + \mu qH(t) \\ & - \mu\int_\Omega \nabla u\nabla u_t dx - \mu\int_\Omega |\nabla u_t|^r\nabla u_t\nabla u dx - \mu\int_{\Gamma_1} f(\cdot,u_t)ud\Gamma \end{aligned} \tag{3.18}$$

Moreover

$$\begin{aligned} & \tfrac{[q-(\gamma+2)][p+2]}{p-\gamma}E_\infty + \tfrac{\eta_0(p+2)(\gamma+2)}{\lambda_2(p-\gamma)}E_\infty - qE_\infty \\ & \qquad = \tfrac{\gamma+2}{\lambda_2(p-\gamma)}(\eta_0(p+2) - \lambda_2(p+2-q))E_\infty \geq 0 \end{aligned}$$

for $\eta_0 \geq \frac{\lambda_2(p+2-q)}{p+2}$. Therefore (3.18) reduces to

$$\begin{aligned} L'(t) \geq & (1-\rho)H^{-\rho}(t)H'(t) + \mu\left[\tfrac{q+2}{2}\right]\|u_t\|^2 + \mu a_1\|\nabla u\|_{\gamma+2}^{\gamma+2} \\ & + \mu\eta_1\|u\|_{p+2}^{p+2} + \mu qH(t) - \mu\int_\Omega \nabla u\nabla u_t dx \\ & - \mu\int_\Omega |\nabla u_t|^r\nabla u_t\nabla u dx - \mu\int_{\Gamma_1} f(\cdot,u_t)ud\Gamma \end{aligned} \tag{3.19}$$

For the sixth term on the right hand side of (3.19), using Hölder inequality and Young's inequality, we have

$$\begin{aligned} \int_\Omega \nabla u\nabla u_t dx & \leq C_2\|\nabla u_t\|_2\|\nabla u\|_{\gamma+2} \\ & \leq C_2\left[\delta_3\|\nabla u_t\|_2^2 + C(\delta_3)\|\nabla u\|_{\gamma+2}^{\gamma+2}\right]\|\nabla u\|_{\gamma+2}^{\frac{-\gamma}{2}} \end{aligned} \tag{3.20}$$

where $C_2 = C_2(\gamma,\Omega)$. Thus, using the estimate (3.5), we obtain

$$\begin{aligned}\int_\Omega \nabla u \nabla u_t dx \leq & \delta_3 (\lambda_2 K_\infty)^{\rho_3} C_2 H^{-\rho}(t) H^{\rho-\rho_3}(0) \|\nabla u_t\|_2^2 \\ & + (\lambda_2 K_\infty)^{\rho_3} C_2 H^{-\rho_3}(0) C(\delta_3) \|\nabla u\|_{\gamma+2}^{\gamma+2} \\ \leq & \delta_3 M_3 H^{-\rho}(t) H^{\rho-\rho_3}(0) \|\nabla u_t\|_2^2 + C(\delta_3) M_3 H^{-\rho}(0) \|\nabla u\|_{\gamma+2}^{\gamma+2}\end{aligned} \tag{3.21}$$

where $M_3 = C_2 (\lambda_2 K_\infty)^{\rho_3}$, $\quad \rho_3 = \frac{\gamma}{2(p+2)}$ and $0 < \rho < \rho_3$.

For the second to the last term on the right hand side of (3.19), using Hölder inequality, we have that

$$\int_\Omega |\nabla u_t|^r \nabla u_t \nabla u \, dx \leq C_3 \left[\|\nabla u_t\|_{r+2}^{r+1} \|\nabla u\|_{\gamma+2}^{\frac{\gamma+2}{r+2}} \right] \|\nabla u\|_{\gamma+2}^{-\frac{\gamma-r}{r+2}} \tag{3.22}$$

where $C_3 = C_3(r,\gamma,\Omega)$ and using Young's inequality together with (3.5), we have

$$\begin{aligned}\int_\Omega |\nabla u_t|^r \nabla u_t \nabla u \, dx \leq & C_3 \left[\delta_4 \|\nabla u_t\|_{r+2}^{r+2} + C(\delta_4) \|\nabla u\|_{\gamma+2}^{\gamma+2} \right] \|\nabla u\|_{\gamma+2}^{-\frac{\gamma-r}{r+2}} \\ \leq & M_4 \delta_4 H^{-\rho}(t) H^{\rho-\rho_4}(0) \|\nabla u_t\|_{r+2}^{r+2} \\ & + C(\delta_4) M_4 H^{-\rho}(0) \|\nabla u\|_{\gamma+2}^{\gamma+2}\end{aligned} \tag{3.23}$$

where $M_4 = C_3 (\lambda_2 K_\infty)^{\rho_4}$, $\rho_4 = \frac{\gamma-r}{(r+2)(p+2)}$ and $0 < \rho < \rho_4$.

To obtain the $L^m(\Gamma_1)$ norm of u for the last term on the right hand side of (3.19), we employ the technique used in [?], by first introducing the Sobolev space of fractional order $W^{s,\gamma+2}(\Omega)$ where $0 < s < 1$ is a parameter to be chosen later. Therefore, using assumption (A_2) and Hölder inequality on Γ_1, we have

$$\int_{\Gamma_1} f(\cdot,u_t) u d\Gamma \leq \|f(\cdot,u_t)\|_{(m+2)',\Gamma_1} \|u\|_{m+2,\Gamma_1} \leq \rho_2 \|u_t\|_{m+2,\Gamma_1}^{m+1} \|u\|_{m+2,\Gamma_1} \tag{3.24}$$

and using the embedding (see [1, Theorem 5.8])

$$\|u\|_{l,\Gamma_1} \leq C \|u\|_{W^{s,\gamma+2}(\Omega)}$$

with $C = C(l,s,\gamma,\Omega) > 0$, that holds for $l \geq 1$ and $s \geq \frac{n}{\gamma+2} - \frac{n-1}{l} > 0$. Then we have,

$$\|u\|_{m+2,\Gamma_1} \leq C_4 \|u\|_{W^{s,\gamma+2}(\Omega)} \tag{3.25}$$

where $C_4 = C_4(m,s,\gamma,\Omega) > 0$ for $0 < s < 1$, and $s \geq \frac{n}{\gamma+2} - \frac{n-1}{m+2}$. Next, using the interpolation (see [7, p. 49]), and Poincaré inequalities (see [19]), we obtain

$$\|u\|_{W^{s,\gamma+2}(\Omega)} \leq C_5 \|u\|_{\gamma+2}^{1-s} \|\nabla u\|_{\gamma+2}^{s} \tag{3.26}$$

for $C_5 = C_5(s,\gamma,\Omega) > 0$. Thus combining (3.25) and (3.26), we have

$$\|u\|_{m+2,\Gamma_1} \leq C_6 \|u\|_{\gamma+2}^{1-s} \|\nabla u\|_{\gamma+2}^{s} \tag{3.27}$$

where $C_6 = C_6(C_4,C_5,m,\gamma,s,\Omega)$. Using Hölder inequality, (3.24) and (3.27), we have

$$\begin{aligned}&\int_{\Gamma_1} f(\cdot,u_t) u d\Gamma \\ &\leq \rho_2 C_6 \|u_t\|_{m+2,\Gamma_1}^{m+1} \|u\|_{p+2}^{1-s} \|\nabla u\|_{\gamma+2}^{s} \\ &\leq \rho_2 C_6 \left(\|u_t\|_{m+2,\Gamma_1}^{m+1} \|u\|_{p+2}^{\frac{p+2}{m+2}\frac{\gamma+2-s(m+2)}{\gamma+2}} \|\nabla u\|_{\gamma+2}^{s} \right) \|u\|_{p+2}^{1-s-\frac{p+2}{m+2}\frac{\gamma+2-s(m+2)}{\gamma+2}}\end{aligned} \tag{3.28}$$

Thus, using Young's inequality, we have

$$\begin{aligned}&\int_{\Gamma_1} f(\cdot,u_t)ud\Gamma \\ &\leq \rho_2 C_6\left[\delta_5\|u_t\|_{m+2,\Gamma_1}^{m+2} + C(\delta_5)\|u\|_{p+2}^{\frac{(p+2)(\gamma+2-s(m+2))}{\gamma+2}}\|\nabla u\|_{\gamma+2}^{s(m+2)}\right]\|u\|_{p+2}^{1-s-\frac{p+2}{m+2}\frac{\gamma+2-s(m+2)}{\gamma+2}}\end{aligned} \tag{3.29}$$

Applying Young's inequality again, we have that for $s < \frac{\gamma+2}{m+2}$,

$$\begin{aligned}\int_{\Gamma_1} f(\cdot,u_t)ud\Gamma \leq \rho_2 C_6\Big(&\delta_5\|u_t\|_{m+2,\Gamma_1}^{m+2} + \delta_6 C(\delta_5)\|\nabla u\|_{\gamma+2}^{\gamma+2} \\ &+ C(\delta_5)C(\delta_6)\|u\|_{p+2}^{p+2}\Big)\|u\|_{p+2}^{(1-s-(p+2))\frac{\gamma+2-s(m+2)}{(m+2)(\gamma+2)}}\end{aligned} \tag{3.30}$$

and for $(1-s-(p+2))\frac{\gamma+2-s(m+2)}{(m+2)(\gamma+2)} < 0$, we have $s < \left(\frac{p-m}{m+2}\right)\left(\frac{\gamma+2}{p-\gamma}\right)$. Therefore, using the estimate (3.5), we obtain

$$\begin{aligned}\int_{\Gamma_1} f(\cdot,u_t)ud\Gamma \leq &\rho_2 M_5\delta_5 H^{\rho-\rho_5}(0)H^{-\rho}(t)\|u_t\|_{m+2,\Gamma_1}^{m+2} + \rho_2 M_5\delta_6 C(\delta_5)H^{-\rho}(0)\|\nabla u\|_{\gamma+2}^{\gamma+2} \\ &+ \rho_2 M_5 C(\delta_5)C(\delta_6)H^{-\rho}(0)\|u\|_{p+2}^{p+2}\end{aligned} \tag{3.31}$$

where we set $M_5 = C_6\left(\frac{\lambda_2}{p+2}\right)^{\rho_5}$, $\quad \rho_5 = \frac{\gamma+2-s(m+2)}{(m+2)(\gamma+2)} - \left(\frac{1-s}{p+2}\right)$ and $0<\rho<\rho_5$.

For $m>0$, since

$$\left(\frac{p-m}{m+2}\right)\left(\frac{\gamma+2}{p-\gamma}\right) \leq \frac{\gamma+2}{m+2} \leq 1$$

It is enough to verify that $m<p$ and

$$\frac{n}{\gamma+2} - \frac{n-1}{m+2} \leq s < \left(\frac{p-m}{m+2}\right)\left(\frac{\gamma+2}{p-\gamma}\right)$$

gives $0 < m < \frac{p(\gamma+2)^2+(p-\gamma)[\gamma(n-1)-2]}{n(p-\gamma)+(\gamma+2)^2}$.

Hence, we choose $\rho \in (0, \min\{\rho_3,\rho_4,\rho_5\})$ such that the inequalities (3.21), (3.23) and (3.31) are satisfied. Now, substituting the estimates (3.21), (3.23) and (3.31) into (3.19), we obtain

$$\begin{aligned}L'(t) \geq &(1-\rho)H'(t)H^{-\rho}(t) + \mu\left[\frac{q+2}{2}\right]\|u_t\|^2 - \mu\delta_3 M_3 H^{-\rho}(t)H^{\rho-\rho_3}(0)\|\nabla u_t\|^2 \\ &- \mu C(\delta_3)M_3 H^{-\rho}(0)\|\nabla u\|_{\gamma+2}^{\gamma+2} - \mu\delta_5 M_5\rho_2 H^{-\rho}(t)H^{\rho-\rho_5}\|u_t\|_{m+2,\Gamma_1}^{m+2} \\ &- \mu\delta_6\rho_2 C(\delta_5)M_5 H^{-\rho}(0)\|\nabla u\|_{\gamma+2}^{\gamma+2} - \mu\rho_2 C(\delta_5)C(\delta_6)M_5 H^{-\rho}(0)\|u\|_{p+2}^{p+2} \\ &- \mu\delta_4 M_4 H^{\rho-\rho_4}(0)H^{-\rho}(t)\|\nabla u_t\|_{r+2}^{r+2} - \mu C(\delta_4)M_4 H^{-\rho}(0)\|\nabla u\|_{\gamma+2}^{\gamma+2} \\ &+ \mu a_1\|\nabla u\|_{\gamma+2}^{\gamma+2} + \mu\eta_1\|u\|_{p+2}^{p+2} + \mu q H(t)\end{aligned}$$

Moreover, using the estimate (3.6), we obtain

$$\begin{aligned}L'(t) \geq &\left[(1-\rho) - \mu\delta_3 M_3 H^{\rho-\rho_3}(0)\right]H^{-\rho}(t)\|\nabla u_t\|^2 \\ &+ \left[(1-\rho) - \mu\delta_4 M_4 H^{\rho-\rho_4}(0)\right]H^{-\rho}(t)\|\nabla u_t\|_{r+2}^{r+2} \\ &+ \left[\rho_1(1-\rho) - \mu\delta_5 M_5\rho_2 H^{\rho-\rho_5}\right]H^{-\rho}(t)\|u_t\|_{m+2,\Gamma_1}^{m+2} + \mu\left[\frac{q+2}{2}\right]\|u_t\|^2 \\ &+ \mu\left[a_1 - \left[C(\delta_3)M_3 + C(\delta_4)M_4 + \delta_6\rho_2 C(\delta_5)M_5\right]H^{-\rho}(0)\right]\|\nabla u\|_{\gamma+2}^{\gamma+2} \\ &+ \mu\left[\eta_1 - \rho_2 C(\delta_5)C(\delta_6)M_5 H^{-\rho}(0)\right]\|u\|_{p+2}^{p+2} + \mu q H(t)\end{aligned} \tag{3.32}$$

Therefore, assume μ in (3.32) to be small enough such that

$$\begin{aligned} &(1-\rho)-\mu\delta_3 M_3 H^{\rho-\rho_3}(0)\geq 0, \quad (1-\rho)-\mu\delta_4 M_4 H^{\rho-\rho_4}(0)\geq 0 \\ &and \quad \rho_1(1-\rho)-\mu\delta_5 M_5\rho_2 H^{\rho-\rho_5}(0)\geq 0. \end{aligned} \tag{3.33}$$

Then, using (3.33) and choosing $\delta_i (i=3,...,6)$ small enough such that $\eta_1 > \rho_2 C(\delta_5)C(\delta_6)M_5 H^{-\rho}(0)$ and $a_1 > \Big([C(\delta_3)M_3 + C(\delta_4)M_4 + \delta_6\rho_2 C(\delta_5)M_5]H^{-\rho}(0)\Big)$. Then, we have that there exist a positive constant C_7 such that (3.32) yields

$$L'(t)\geq \mu C_7\left(\|u_t\|^2+\|\nabla u\|_{\gamma+2}^{\gamma+2}+H(t)\right) \tag{3.34}$$

where $C_7 := \min\Big\{q,\ \frac{(q+2)}{2},\ M_6,\ [a_1 - M_7]\Big\}$, where $M_6 = [\eta_1 - \rho_2 C(\delta_5)C(\delta_6)M_5 H^{-\rho}(0)]$ and $M_7 = [C(\delta_3)M_3 + C(\delta_4)M_4 + \delta_6\rho_2 C(\delta_5)M_5]H^{-\rho}(0)$. Therefore, choosing

$$L(0) = H^{1-\rho}(0) + \mu\int_\Omega u_0 u_1 dx > 0$$

then from (3.34), we have that $L(t)$ is an increasing function for $t\geq 0$, satisfying

$$L(t)\geq L(0) > 0 \quad \forall t\geq 0.$$

On the other hand, we have

$$L^{\frac{1}{1-\rho}}(t)\leq 2^{\frac{1}{1-\rho}}\left[H(t)+\mu^{\frac{1}{1-\rho}}\left(\int_\Omega u_t u dx\right)^{\frac{1}{1-\rho}}\right] \tag{3.35}$$

Now, using Hölder inequality, we get

$$\left|\int_\Omega u u_t dx\right|\leq C_8\|u\|_{p+2}\|u_t\|_2\leq KC_8\|\nabla u\|_{\gamma+2}\|u_t\|_2$$

where $C_8 = C_8(p,\Omega)$. Then by Young's inequality, we have

$$\left|\int_\Omega u u_t dx\right|^{\frac{1}{1-\rho}}\leq C_9\left[\|\nabla u\|_{\gamma+2}^{\frac{\omega}{1-\rho}}+\|u_t\|^{\frac{\theta}{1-\rho}}\right] \tag{3.36}$$

where $C_9 = C_9(C_8, K, \omega, \theta, \rho)$ and where $\frac{1}{\omega}+\frac{1}{\theta} = 1$. Now choosing $\theta = 2(1-\rho)$ and setting $\frac{\omega}{1-\rho} = \frac{2}{1-2\rho}\leq \gamma+2$, so that $\rho\leq\frac{\gamma}{2(\gamma+2)}$, then (3.36) yields

$$\left|\int_\Omega u u_t dx\right|^{\frac{1}{1-\rho}}\leq C_9\left[\|\nabla u\|_{\gamma+2}^{\gamma+2}+\|u_t\|^2\right] \tag{3.37}$$

Combining the choice of ρ in (3.37), with the previous choices, we choose $0<\rho\leq\min\{\rho_3,\rho_4,\rho_5,\frac{\gamma}{2(\gamma+2)}\}$. Thus, from (3.35) and (3.37), we have

$$L^{\frac{1}{1-\rho}}(t)\leq C_{10}\left[\|u_t\|^2+\|\nabla u\|_{\gamma+2}^{\gamma+2}+H(t)\right] \tag{3.38}$$

where $C_{10} = C_{10}(C_9,\mu,\rho)$. Therefore, using the estimates (3.34) and (3.38), we have that there exist a positive constant $C_{11} = C_{11}(\mu, C_7, C_{10})$ such that

$$L'(t)\geq C_{11}L^{\frac{1}{1-\rho}}(t) \quad \forall t\geq 0. \tag{3.39}$$

Dividing both sides of (3.39) by $L^{\frac{1}{1-\rho}}(t)$ and applying a simple integration gives

$$L^{\frac{\rho}{1-\rho}}(t)\geq\left[L^{-\frac{\rho}{1-\rho}}(0)-C_{11}\frac{\rho}{1-\rho}t\right]^{-1}$$

Thus $L(t)$ blow up in time

$$T^* \leq [C_{11}\frac{\rho}{1-\rho}]^{-1}L^{-\frac{\rho}{1-\rho}}(0)$$

Remark 3.1. *When the nonlinear terms take the form; $f(x,s) = c(x)|s|^m s$ and $g(x,s) = d(x)|s|^p s$, where $c \in C^0(\Gamma_1)$ and $d \in C^0(\Omega)$ are smooth and bounded functions with positive values, the results of Theorem 3.1 hold provided*

$$\rho_1 \leq c(x) \leq \rho_2 \text{ and } \lambda_1 \leq d(x) \leq \lambda_2$$

References

[1] R. A. Adams, Sobolev spaces (Academic Press, New York, 1975).

[2] M. M. Cavalcanti, V. N. Domingos Cavalcanti, and P. Martinez, Existence and decay rate estimates for the wave equation with nonlinear boundary damping and source term, Journal of Differential Equations 203 (2004), no. 1, 119-158.

[3] V. Georgiev and G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source term, J. Differential Equations **109**, (1994), 295-308.

[4] J. Jeong, J. Park, and Y. H. Kang, Global nonexistence of solutions for a quasilinear wave equation with acoustic boundary conditions, Boundary Value Problems. **42**, (2017), 1.

[5] H. A. Levine, Instability and nonexistence of global solutions to nonlinear wave equations of the form, Trans. Amer. Math. Soc. **192**, (1974) , 1-21.

[6] H. A. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equations, SIAM J. Math. Anal. **5**, (1974), 138-146.

[7] J. L. Lions and E. Magenes, Problémes aux limites non homogénes, Vol. 1 (Dunod, Paris, 1968).

[8] W. Liu and M. Wang, Global nonexistence of solutions with positive initial energy for a class of wave equations, Math. Meth. Appl. Sci. **32**, (2009), 594-605.

[9] S.A. Messaoudi, Blow up of positive initial energy solutions of a nonlinear viscoelastic hyperbolic equation, Journal of Mathematical Analysis and Application, **320**, (2006), 902 -915.

[10] S.A. Messaoudi and B. Said-Houari, Global nonexistence of solutions of a class of wave equations with nonlinear damping and source terms, Math. Methods. Appl. Sci. **27**, (2004), 1687-1696.

[11] S.A. Messaoudi, A. A. Talahmeh, On wave equation: review and recent results. Arab. J. Math. **7**, (2018), 113-145.

[12] P. A. Ogbiyele, Global existence and blow up of positive initial energy solution of a quasi-linear wave equation with nonlinear damping and source terms, Int. J. Dynamical Systems and Differential Equations, **10**(4), (2020), 299-320.

[13] P. A. Ogbiyele, On global existence and nonexistence of solutions to a quasi-linear wave equation with memory, nonlinear damping and source terms. submitted

[14] E. Piskin, On the decay and blow up of solutions for a quasilinear hyperbolic equations with nonlinear damping and source terms, Bound. Value Probl. **127**, (2015), 1.

[15] F. Sun and M. Wang, Global solutions for a nonlinear hyperbolic equation with boundary memory source term, J. Inequalities and Applications, (2006), 1-16.

[16] E. Vitillaro, Some new results on global nonexistence and blow-up for evolution problems with positive initial energy, Rend. Ist. Mat. Univ. Trieste **31** Suppl. **2** (2000), 245-275.

[17] E. Vitillaro, Global existence for the wave equation with nonlinear boundary damping and source terms, J. Differential Equations **186**, (2002) 259-298.

[18] Z. Yang, Blowup of solutions for a class of nonlinear evolution equations with nonlinear damping and source terms, Math. Methods Appl. Sci. **25**, (2002), 825 -833.

[19] W. P. Ziemer, Weakly differentiable functions, Graduate Text in Mathematics, Vol. 120 (Springer-Verlag), (1989) .