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On asymptotic behavior of solution to a nonlinear wave equation with Space-time speed of propagation and damping terms

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Abstract

In this paper, we consider the asymptotic behavior of solution to the nonlinear damped wave equation

$$u_{tt} - div\big(a(t,x)\nabla u\big) + b(t,x)u_t = -|u|^{p-1}u \quad t \in [0,\infty), \quad x \in \mathbf{R}^n$$

$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x) \qquad x \in \mathbf{R}^n$$

with space-time speed of propagation and damping potential. We obtained L^2 decay estimates via the weighted energy method and under certain suitable assumptions on the functions $a(t,x)$ and $b(t,x)$. The technique follows that of Lin et al.[8] with modification to the region of consideration in $\mathbf{R}^n$. These decay result extends the results in the literature.

Subjclass Primary: *35L05, 35L70; Secondary: 37L15*

Keywords: *Space-time speed of propagation, Space-time dependent damping, Asymptotic behavior, Weighted energy method.*

1. Introduction

In this paper, we are concerned with the asymptotic behavior of solution to the following nonlinear wave equation

$$\begin{cases} u_{tt} - \operatorname{div}\Big(a(t,x)\nabla u\Big) + b(t,x)u_t = -|u|^{p-1}u, & t \in [0,\infty), \quad x \in \mathbf{R}^n \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x) & x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

with space-time dependent coefficients of the form

$$b(t,x) = b_0(1+|x|^2)^{\frac{-\alpha}{2}}(1+t)^{-\beta} \tag{1.2}$$

and

$$\rho_1(1+|x|^2)^{\frac{\delta}{2}}(1+t)^{\gamma}|\xi|^2 \leq a(t,x)\xi\cdot\xi \leq \rho_0(1+|x|^2)^{\frac{\delta}{2}}(1+t)^{\gamma}|\xi|^2, \quad \xi \in \mathbf{R}^n \tag{1.3}$$

where $a(t,x) = \eta(t)^{-1}\rho(x)$ and $\eta(t) = (1+t)^{-\gamma}$. In addition, $b_0 > 0$, $\rho_0 > 0$, $\alpha + \delta \in [0,2)$ and $\beta + \gamma \in [0,1)$, where $u = u(t,x)$. More precisely, $\alpha + \beta + \delta + \gamma \in [0,1)$. Equations of the form (1.1) arise in the study of nonlinear wave equations describing the motion of body traveling in an in-homogeneous medium. They appear in various aspects of Mathematical Physics, Geophysics and Ocean acoustics.

In the case of scalar coefficients and bounded smooth domains Ω, there is an extensive literature on energy dacay results. For the semi-linear wave equation

$$u_{tt} - \Delta u + u_t = |u|^p, \tag{1.4}$$

Todorova and Yordanov [18] showed that $C_n = 1 + \frac{2}{n}$ is the critical exponent(Fujita exponent) for $p < \infty$ $(n < 3)$ and $p < 1 + \frac{2}{n}(n \geq 3)$.

Nishihara in his paper [11] showed that the decay rate of solution to the damped linear wave equation follows that of self similar solution of its corresponding heat equation for $n = 3$ and showed this by obtaining $L^p - L^q$ estimates on their difference. For similar results on 1-dimension and 2-dimensions, see Marcati and Nishihara [9] and Hosono and Ogawa [5] respectively, and in any dimension, see Narazaki [10]. Hence, it is expected that the behavior of the solution to equation (1.4) is similar to that of the corresponding heat equation

$$u_t - \Delta u = |u|^p, \tag{1.5}$$

whose similarity solution $u_a(t,x)$ has the form $t^{\frac{-1}{p-1}}F(xt^{-\frac{1}{2}})$ with $a = \lim_{|x|\to\infty} |x|^{\frac{2}{p-1}} f(x) \geq 0$ provided that $p < 1 + \frac{2}{n}$.

In the case of time dependent potential type of damping, with equations of the form

$$u_{tt} - \Delta u + b(t)u_t + |u|^{p-1}u = 0, \tag{1.6}$$

there are also several results on the decay rate of the solution. Nishihara and Zhai [13], used a weighted energy method similar to those in [18] and obtained decay estimates of the form

$$\begin{aligned} \|u\|_2 &\leq Ct^{-(\frac{n}{4(p-1)})(1+\beta)} \\ \|u\|_1 &\leq Ct^{-(\frac{n}{2(p-1)})(1+\beta)} \end{aligned} \tag{1.7}$$

under the assumption that $b(t) \approx (1+t)^{-\beta}$. For Cauchy problem of the form

$$u_{tt} - a^2(t)\Delta u + b(t)u_t + c_0|u|^{p-1}u = 0, \tag{1.8}$$

it is well known that the interplay between the coefficient $a^2(t)$ and the term $b(t)u_t$ induces different effect on the asymptotic behavior of the energy $E(t)$ given by

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{a^2(t)}{2}\|\nabla u\|_2^2 + \frac{1}{p}\|u\|_p^p. \tag{1.9}$$

For more details see [2, 3, 4, 20] and the references therein. In [1] Bui considered the asymptotic behavior of the nonlinear problem (1.8) with $a(t) = (1+t)^{\ell}$ and $b(t) = \mu(1+\ell)(1+t)^{-1}$, $\ell > 0$, $c_0 = 0$ and obtained the following estimate

$$\|u_t(t,\cdot),(1+t)^{\ell}\nabla u(t,\cdot)\|_{L^2} \leq (1+t)^{\ell+(\ell+1)\max\{\mu^*-\frac{1}{2},-1\}}\left(\|u_1\|_{H^1}+\|u_2\|_{L^2}\right) \tag{1.10}$$

with $\mu^* = \frac{1}{2}(1-\mu-\frac{\ell}{\ell+1})$.

In the case of damped wave equation with space dependent potential type of damping;

$$u_{tt} - \Delta u + b(x)u_t + |u|^{p-1}u = 0, \tag{1.11}$$

where $b_1(1+|x|)^{-\alpha} \leq b(x) \leq b_2(1+|x|)^{-\alpha}$ and $b_1, b_2 > 0$, Todorova and Yordanov [19] investigated the decay rate of the energy when $0 \leq \alpha < 1$. They obtained several decay rate types for solutions of (1.11) depending on p and α. These decay rates take the form

$$\Big(\|u_t\|_2 + \|\nabla u\|_2, \|u\|_{p+1}\Big) = O\Big(t^{\frac{-1}{p-1}+\delta}, t^{-\frac{p+1}{2(p-1)}+\delta}\Big) \tag{1.12}$$

if $1 < p < 1 + \frac{2\alpha}{n-\alpha}$ and

$$\Big(\|u_t\|_2+\|\nabla u\|_2, \|u\|_{p+1}\Big) = O\Big(t^{-(1+\frac{\alpha}{2})\frac{1}{p-1}+\frac{n}{2(p+1)}+\delta}, t^{-(1+\frac{\alpha}{2})\frac{p+1}{2(p-1)}+\frac{n}{4}+\delta}\Big) \tag{1.13}$$

if $1 + \frac{2\alpha}{n-\alpha} < p < 1 + \frac{2(4-\alpha)}{(n-\alpha)(4-\alpha)}$, for $t > 1$, where δ is a constant. Nishihara[12] also considered the asymptotic behavior of solution to the semi-linear wave equation (1.11) with $b(x)$ satisfying

$$b_1(1 + |x|^2)^{-\frac{\alpha}{2}} \le b(x) \le b_2(1 + |x|^2)^{-\frac{\alpha}{2}} \tag{1.14}$$

and obtained decay rates of the following type

$$\|u(t,\cdot)\|_2 \le \begin{cases} C(1+t)^{-\frac{n-2\alpha}{2(2-\alpha)}} & \text{if } 1+\frac{2}{n-\alpha} \le p < \frac{n+2}{n-2} \\ C(1+t)^{-\frac{2}{2-\alpha}(\frac{1}{p-1})-\frac{n}{4}} & \text{if } 1+\frac{2\alpha}{n-\alpha} < p \le 1+\frac{2}{n-\alpha} \\ C(1+t)^{-\frac{2}{2-\alpha}(\frac{1}{p-1})-\frac{n}{4}}[\log(t+2)]^{\frac{1}{2}} & \text{if } p = 1+\frac{2\alpha}{n-\alpha} \\ C(1+t)^{-\frac{1}{p-1}+\frac{\alpha}{2(2-\alpha)}} & \text{if } 1 < p < 1+\frac{2\alpha}{n-\alpha} \end{cases} \tag{1.15}$$

where $\alpha \in [0, 1)$.
Ikehata and Inoue [6] studied nonlinear wave equations with $b(x) = b_0(1 + |x|)^{-1}$ and showed that solutions to (1.11) depend on the coefficient b_0 and their decay estimate takes the form

$$\|u\| = O(t^{-1+\mu}) \qquad \|u_t\|_2^2 + \|\nabla u\|_2^2 = O(t^{-1+\mu}) \tag{1.16}$$

where
$1 < \mu + b_0 < 1 + b_0$ if $0 < b_0 \le 1$
$0 \le \mu < 1$ if $b_0 \ge 1$.

Moreover, for damped wave equations with space-time dependent potential type of damping

$$\begin{aligned} &u_{tt} - \Delta u + b(t,x)u_t + |u|^{p-1}u = 0, \quad t > 0,\ x \in \mathbf{R}^n \\ &u(0,x) = u_0(x), u_t(0,x) = u_1(x), \quad x \in \mathbf{R}^n, \end{aligned} \tag{1.17}$$

Lin et al. [8] considered decay rates of solution to (1.17) and showed using the weighted energy method that the L^2 norm of the solution decays as

$$\|u(t.\cdot)\|_2 \leq \begin{cases} C(1+t)^{-(\frac{1}{p-1}-\frac{\alpha}{2(2-\alpha)})(1+\beta)} & \text{if } \frac{\alpha(p+1)}{p-1} > n \\ C(1+t)^{-(\frac{1}{p-1}-\frac{\alpha}{2(2-\alpha)})(1+\beta)}\log(t+2), & \text{if } \frac{\alpha(p+1)}{p-1} = n \\ C(1+t)^{-(1+\beta)\frac{1}{p-1}+\frac{1+\beta}{2(2-\alpha)}(N-\alpha\frac{2}{p-1})} & \text{if } \frac{\alpha(p+1)}{p-1} < n \end{cases}$$

(1.18)

For nonlinear wave equations with variable coefficients which exhibit a dissipative term with a space dependent potential

$$(1.19) \qquad u_{tt} - \nabla\cdot(b(x)\nabla u) + \nabla\cdot(b(x)u_t) = 0, x \in \mathbf{R}^n, \quad t > 0$$

under the assumption that

$$(1.20) \qquad b_0(1+|x|)^\beta|\xi|^2 \leq b(x)\xi\cdot\xi \leq b_1(1+|x|)^\beta|\xi|^2, \quad \xi \in \mathbf{R}^n,$$

where $b_0 > 0$, $b_1 > 0$ and $\beta \in [0,2)$. R. Ikehata et al. [7] obtained long time asymptotics for solutions to (1.19)-(1.20) as a combination of solutions of wave and diffusion equations under certain assumptions on b in an exterior domain, see also [15].

Said-Houari [17] considered a viscoelastic wave equation with space-time dependent damping potential and an absorbing term

$$\begin{aligned} &u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + b(t,x)u_t + |u|^{p-1}u = 0, \quad t > 0, \quad x \in \mathbf{R}^n \\ &u(0,x) = u_0(x), u_t(0,x) = u_1(x) \quad x \in \mathbf{R}^n \end{aligned}$$

(1.21)

and by using a weighted energy method, they showed that the L^2 decay rates are the same as those in [8].

More recently, Roberts[16] under the assumption that

$$b_0(1+|x|)^\beta \leq b(x) \leq b_1(1+|x|)^\beta \quad \text{and} \quad a_0(1+|x|)^{-\alpha} \leq a(x) \leq a_1(1+|x|)^{-\alpha}$$

with

$$(1.22) \qquad \alpha < 1, \quad 0 \leq \beta < 2, \quad 2\alpha+\beta < 2,$$

obtained energy decay estimates of solution to the dissipative non-linear wave equation

$$(1.23) \quad \begin{aligned} &u_{tt} - \operatorname{div}(b(x)\nabla u) + a(x)u_t + |u|^{p-1}u = 0, \quad x \in \mathbf{R}^n, \quad t > 0 \\ &u(0,x) = u_0(x) \in H^1(\mathbf{R}^n), \quad u_t(0,x) = u_1(x) \in L^2(\mathbf{R}^n), \end{aligned}$$

using a modification of the weighted multiplier technique introduced by Todorova and Yordanov[14].

In this paper, by using the weighted L^2-energy method similar to that of [8], we obtain decay estimates of the energy of the solution to (1.1), where $a(t,x)$ and $b(t,x)$ have the form in (1.2)-(1.3) above. In [8], the space $\mathbf{R}^n$ was divided into two zones

$$Z(t;L,t_0) := \{x \in \mathbf{R}^n | (t_0+t)^2 \geq L + |x|^2\}$$

and $Z^c(t;L,t_0) = \mathbf{R}^n \backslash Z(t;L,t_0)$. To obtain boundedness on certain estimates on Z, a further division of Z was required. Here, we split the domain into two zones

$\Omega(t,L,t_0) = \{x \in \mathbf{R}^n : (t_0+t)^A \geq L + |x|^2\}$ and

$\Omega^c(t,L,t_0) = \mathbf{R}^n \backslash \Omega(t,L,t_0)$

which depend on the weighted function for $A = \frac{2(1+\beta+\gamma)}{2-(\alpha+\delta)}$ and positive constants L, t_0. With this choice, we overcome the challenge of splitting the first zone in order to obtain boundedness for every estimate on $\Omega(t;L,t_0)$ in the proof.

2. Preliminaries

In this section, we state some basic assumptions used in this paper. First, we introduce the following notations. $L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$, the Lebesgue space with norm $\|\cdot\|_p$ and $H^1_\rho(\mathbf{R}^n)$ the Sobolev space defined by

$$H^1_\rho(\mathbf{R}^n) := \{u \in L^{\frac{2n}{n-2+\delta}} : \int_{\mathbf{R}^n} (1+|x|^2)^{\frac{\delta}{2}} |\nabla u|^2 dx < \infty\}. \tag{2.1}$$

Lemma 2.1. *(Caffarelli-Kohn-Nirenberg)*
There exist a constant $C > 0$ such that the inequality

$$\||x|^\sigma u\|_{L^r} \leq C \||x|^\delta \nabla u\|^\theta_{L^q} \||x|^\ell u\|^{1-\theta}_{L^p} \tag{2.2}$$

holds for all $u \in C_0^\infty(\mathbf{R}^n)$ if and only if the following relations hold:

$$\frac{1}{r} + \frac{\sigma}{n} = \theta\Big(\frac{1}{q} + \frac{\delta-1}{n}\Big) + (1-\theta)\Big(\frac{1}{p} + \frac{\ell}{n}\Big) \tag{2.3}$$

with $p,q \geq 1$. $r > 0$, $0 \leq \theta \leq 1$. $\delta - d \leq 1$ if $\theta > 0$ and $\frac{1}{p} + \frac{\delta-1}{n} = \frac{1}{r} + \frac{\sigma}{n}$

Remark 1. *When $\sigma = \delta = \ell = 0$, the Lemma is referred to as the Gagliardo-Nirenberg inequality.*

We define the weighted function $\psi(t,x)$ as follows:

$$\psi(t,x) = \lambda \frac{(L+|x|^2)^{\frac{2-(\alpha+\delta)}{2}}}{(t_0+t)^{1+\beta+\gamma}} \tag{2.4}$$

for a small positive constant $\lambda = \frac{b_0(1+\beta+\gamma)}{2\rho_0(2-(\alpha+\delta))^2}$ and $t_0 \geq L \geq 1$. Moreover, we have

$$\begin{aligned} \psi_t(t,x) = & \quad -\lambda(1+\beta+\gamma)\frac{(L+|x|^2)^{\frac{2-(\alpha+\delta)}{2}}}{(t_0+t)^{2+\beta+\gamma}} \\ \nabla\psi(t,x) = & \quad \lambda(2-(\alpha+\delta))\frac{(L+|x|^2)^{\frac{-\alpha-\delta}{2}}x}{(t_0+t)^{1+\beta+\gamma}} \\ |\nabla\psi(t,x)|^2 = & \quad \lambda^2(2-(\alpha+\delta))^2\frac{(L+|x|^2)^{-\alpha-\delta}|x|^2}{(t_0+t)^{2+2\beta+2\gamma}} \end{aligned}$$

and consequently, we have

$$\frac{a(t,x)|\nabla\psi|^2}{(-\psi_t(t,x))} \leq \frac{1}{2}b(t,x). \tag{2.5}$$

In the sequel, we will denote the function $\psi(t,x)$ by ψ for simplicity. To begin, we state the following lemmas which will be needed in the proof of the main result. First, we define the functions $\mathcal{E}(t)$ and $\mathcal{H}(t)$ associated to problem (1.1) by

$$\mathcal{E}(t) := e^{2\psi}\eta(t)\Big[\tfrac{1}{2}|u_t|^2 + \tfrac{a(t,x)}{2}|\nabla u|^2 + \tfrac{1}{p+1}|u|^{p+1}\Big] \tag{2.6}$$

and

$$\mathcal{H}(t) := e^{2\psi}\eta(t)\Big[uu_t + \frac{b(t,x)}{2}|u|^2\Big] \tag{2.7}$$

respectively. Then for the function $\mathcal{E}(t)$ in (2.6), we have the following result.

Lemma 2.2. *Let u be a solution of (1.1), then the function $\mathcal{E}(t)$ defined in (2.6), satisfies*

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) \leq & \quad \nabla\cdot(e^{2\psi}\rho(x)\nabla u u_t) + e^{2\psi}\eta(t)\Big[-\tfrac{b(t,x)}{4} + \psi_t\Big]|u_t|^2 + e^{2\psi}\tfrac{\eta_t(t)}{2}|u_t|^2 \\ & + e^{2\psi}\eta(t)\Big[\tfrac{-\gamma}{(p+1)(1+t)} + \tfrac{2\psi_t}{p+1}\Big]|u|^{p+1} + e^{2\psi}\Big[\tfrac{\rho(x)\psi_t}{3}\Big]|\nabla u|^2. \end{aligned} \tag{2.8}$$

Proof. Multiplying (1.1) by $e^{2\psi}u_t$ and using (2.5), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left[e^{2\psi}\left[\frac{1}{2}|u_t|^2+\frac{a(t,x)}{2}|\nabla u|^2+\frac{1}{p+1}|u|^{p+1}\right]\right]\\
&=\nabla\cdot(e^{2\psi}a(t,x)\nabla u u_t)+e^{2\psi}\Big[\psi_t-b(t,x)\Big]|u_t|^2+\frac{e^{2\psi}a_t(t,x)}{2}|\nabla u|^2\\
&+\frac{e^{2\psi}a(t,x)}{\psi_t}\Big[\psi_t|\nabla u|^2-\nabla\psi u_t\Big]^2-\frac{e^{2\psi}a(t,x)|\nabla\psi|^2}{\psi_t}|u_t|^2+\frac{2e^{2\psi}\psi_t}{p+1}|u|^{p+1}\\
&\le\nabla\cdot(e^{2\psi}a(t,x)\nabla u u_t)+e^{2\psi}\Big[\psi_t-\frac{1}{2}b(t,x)\Big]|u_t|^2+\frac{e^{2\psi}a_t(t,x)}{2}|\nabla u|^2\\
&+\frac{e^{2\psi}a(t,x)}{\psi_t}\Big[\psi_t|\nabla u|-\nabla\psi u_t\Big]^2+\frac{2e^{2\psi}\psi_t}{p+1}|u|^{p+1},
\end{aligned}
\tag{2.9}
$$

where we have used

$$e^{2\psi}u_t\cdot b(t,x)u_t=e^{2\psi}b(t,x)|u_t|^2. \tag{2.10}$$

By employing Schwartz inequality, we observe that

$$
\begin{aligned}
\frac{e^{2\psi}a(t,x)}{\psi_t}\quad&\Big[\psi_t|\nabla u|-\nabla\psi u_t\Big]^2\\
&=\frac{e^{2\psi}a(t,x)}{\psi_t}\Big[|\psi_t|^2|\nabla u|^2-2\psi_t u_t\nabla u\cdot\nabla\psi+|\nabla\psi|^2|u_t|^2\Big]\\
&\le\frac{e^{2\psi}a(t,x)}{\psi_t}\Big[\frac{1}{3}|\psi_t|^2|\nabla u|^2-\frac{1}{2}|\nabla\psi|^2|u_t|^2\Big].
\end{aligned}
\tag{2.11}
$$

Hence, using (2.5) in (2.11) and substituting the resulting estimate in (2.9), we obtain

$$
\begin{aligned}
\frac{d}{dt}\quad&\left[e^{2\psi}\left[\frac{1}{2}|u_t|^2+\frac{a(t,x)}{2}|\nabla u|^2+\frac{1}{p+1}|u|^{p+1}\right]\right]\\
\le\quad&\nabla\cdot(e^{2\psi}a(t,x)\nabla u u_t)+e^{2\psi}\Big[\psi_t-\frac{b(t,x)}{4}\Big]|u_t|^2+\frac{2e^{2\psi}\psi_t}{p+1}|u|^{p+1}\\
&+e^{2\psi}\Big[\frac{a_t(t,x)}{2}+\frac{a(t,x)\psi_t}{3}\Big]|\nabla u|^2
\end{aligned}
\tag{2.12}
$$

and multiplying (2.12) by $\eta(t)$, we get

$$
\begin{aligned}
\frac{d}{dt}\quad&\left[e^{2\psi}\eta(t)\left[\frac{1}{2}|u_t|^2+\frac{a(t,x)}{2}|\nabla u|^2+\frac{1}{p+1}|u|^{p+1}\right]\right]\\
\le\quad&\nabla\cdot(e^{2\psi}\rho(x)\nabla u u_t)+e^{2\psi}\eta(t)\Big[-\frac{b(t,x)}{4}+\psi_t\Big]|u_t|^2+e^{2\psi}\frac{\eta_t(t)}{2}|u_t|^2\\
&+e^{2\psi}\eta(t)\Big[\frac{-\gamma}{(p+1)(1+t)}+\frac{2\psi_t}{p+1}\Big]|u|^{p+1}+e^{2\psi}\left[\frac{\rho(x)\psi_t}{3}\right]|\nabla u|^2.
\end{aligned}
\tag{2.13}
$$

□

Now, for the function $\mathcal{H}(t)$, we have the following lemma.

Lemma 2.3. *Let u be a solution of (1.1), then the function $\mathcal{H}(t)$ defined in (2.7), satisfies*

$$\begin{aligned}\frac{d}{dt}\mathcal{H}(t) \leq\ & \nabla\cdot(e^{2\psi}\rho(x)u\nabla u) + e^{2\psi}\eta(t)|u_t|^2 + 2e^{2\psi}\eta(t)\psi_t u u_t - e^{2\psi}\eta(t)|u|^{p+1}\\ & -\frac{e^{2\psi}\rho(x)}{4}|\nabla u|^2 + e^{2\psi}\eta(t)\left[\frac{b_t(t,x)}{2} + \frac{b(t,x)\psi_t}{3}\right]|u|^2\\ & + e^{2\psi}\frac{\eta_t(t)b(t,x)}{2}|u|^2 + e^{2\psi}\eta_t(t)uu_t \end{aligned} \tag{2.14}$$

Proof. Multiplying (1.1) by $e^{2\psi}u$ and using the estimate (2.5), we get

$$\begin{aligned} & \frac{d}{dt}\left[e^{2\psi}\left[uu_t + \frac{b(t,x)}{2}|u|^2\right]\right]\\ = & \ \nabla\cdot(e^{2\psi}a(t,x)u\nabla u) + e^{2\psi}|u_t|^2 + 2e^{2\psi}\psi_t uu_t + e^{2\psi}\frac{b_t(t,x)}{2}|u|^2\\ & -e^{2\psi}a(t,x)|\nabla u|^2 - \frac{a^2(t,x)|\nabla\psi|^2}{\psi_t b(t,x)}|\nabla u|^2 e^{2\psi} - e^{2\psi}|u|^{p+1}\\ & +\frac{b(t,x)}{\psi_t}\left[|\psi_t u + \frac{a(t,x)\nabla\psi}{b(t,x)}|\nabla u|\right]^2 e^{2\psi}\\ \leq & \ \nabla\cdot(e^{2\psi}a(t,x)u\nabla u) + e^{2\psi}|u_t|^2 + 2e^{2\psi}\psi_t uu_t + e^{2\psi}\frac{b_t(t,x)}{2}|u|^2\\ & -\frac{e^{2\psi}a(t,x)}{2}|\nabla u|^2 + \frac{b(t,x)}{\psi_t}\left[|\psi_t u - \frac{a(t,x)\nabla\psi}{b(t,x)}|\nabla u|\right]^2 e^{2\psi} - e^{2\psi}|u|^{p+1} \end{aligned} \tag{2.15}$$

where we have used

$$\begin{aligned} e^{2\psi}b(t,x)uu_t = & \ \frac{d}{dt}\left[\frac{e^{2\psi}b(t,x)}{2}|u|^2\right] - e^{2\psi}\psi_t b(t,x)|u|^2\\ & -e^{2\psi}\frac{b_t(t,x)}{2}|u|^2. \end{aligned} \tag{2.16}$$

Using Schwartz inequality for the second to the last term on the right hand side of (2.15), we have the following estimate

$$\begin{aligned} & \frac{b(t,x)}{\psi_t}\left[|\psi_t u + \frac{a(t,x)\nabla\psi}{b(t,x)}|\nabla u|\right]^2\\ \leq & \ \frac{b(t,x)}{\psi_t}\left[\frac{1}{3}|\psi_t|^2|u|^2 - \frac{|a(t,x)|^2|\nabla\psi|^2}{2|b(t,x)|^2}|\nabla u|^2\right]. \end{aligned} \tag{2.17}$$

In a similar way, using (2.5) in (2.17), and substituting the resulting estimate in (2.15), we get

$$\begin{aligned} & \frac{d}{dt}\left[e^{2\psi}\left[uu_t + \frac{b(t,x)}{2}|u|^2\right]\right]\\ \leq & \ \nabla\cdot(e^{2\psi}a(t,x)u\nabla u) + e^{2\psi}|u_t|^2 + 2e^{2\psi}\psi_t uu_t + e^{2\psi}\frac{b_t(t,x)}{2}|u|^2\\ & -\frac{e^{2\psi}a(t,x)}{4}|\nabla u|^2 + e^{2\psi}\frac{b(t,x)\psi_t}{3}|u|^2 - e^{2\psi}|u|^{p+1} \end{aligned} \tag{2.18}$$

and multiplying (2.18) by $\eta(t)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[e^{2\psi}\eta(t)\left[uu_t + \frac{b(t,x)}{2}|u|^2\right]\right] \\
&\leq \nabla\cdot(e^{2\psi}\rho(x)u\nabla u) + e^{2\psi}\eta(t)|u_t|^2 + 2e^{2\psi}\eta(t)\psi_t uu_t - e^{2\psi}\eta(t)|u|^{p+1} \\
&-\frac{e^{2\psi}\rho(x)}{4}|\nabla u|^2 + e^{2\psi}\eta(t)\left[\frac{b_t(t,x)}{2} + \frac{b(t,x)\psi_t}{3}\right]|u|^2 \\
&+e^{2\psi}\frac{\eta_t(t)b(t,x)}{2}|u|^2 + e^{2\psi}\eta_t(t)uu_t.
\end{aligned} \tag{2.19}$$

□

3. Main result

In this section, we consider the long time behavior of the solution to (1.1). The result here is obtained via a weighted energy method and the technique follows that of Lin et al.[8]. For local existence result, the compactness condition on the support of the initial data is replaced by the following condition:

$$\begin{aligned}
I_0 := &\int\limits_{\Omega(0;L,t_0)} \eta(0)\left[t_0^{\beta+\frac{\alpha A}{2}}\left[|u_1|^2 + a(0,x)|\nabla u_0|^2\right] + b(0,x)|u_0|^2\right]e^{2\psi(0,x)}dx \\
&+ \int\limits_{\Omega^c(0;L,t_0)} \eta(0)\left[(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\left[|u_1|^2 + a(0,x)|\nabla u_0|^2\right] + b(0,x)|u_0|^2\right] \\
&e^{2\psi(0,x)}dx < +\infty.
\end{aligned} \tag{3.1}$$

With respect to the size of $(1+|x|^2)$ and $(1+t)$ and considering the weighted function ψ, we partition the space $\mathbf{R}^n$ into the following zones:

$$\Omega(t,L,t_0) = \{x\in\mathbf{R}^n : (t_0+t)^A \geq L+|x|^2\} \quad \text{and}$$
$$\Omega^c(t,L,t_0) = \mathbf{R}^n\backslash\Omega(t,L,t_0)$$

which is a modification of the zones as inspired by Lin et. al. [8], where $A = \frac{2(1+\beta+\gamma)}{2-(\alpha+\delta)}$. Since $\alpha+\beta+\delta+\gamma\in[0,1)$, it follows that $A<2$.

Theorem 3.1. *Let u be the solution of (1.1) and let $a(t,x)$, $b(t,x)$ satisfy (1.2) and (1.3) for $2 < p+1 < \frac{2n}{n-2+\delta}$ when $n\geq 2$. Suppose that $(u_0,u_1)\in H^1_\rho(\mathbf{R}^n)\cap L^2(\mathbf{R}^n)$ and (**??**) holds. Then there exist a unique solution u of (1.1) with $u\in L^\infty([0,\infty);H^1_\rho(\mathbf{R}^n))$ and $u_t\in L^\infty([0,\infty);L^2(\mathbf{R}^n))$ which satisfies the following estimate*

$$(3.2)\|u\|^2_{L_2} \le \begin{cases} C(1+t)^{-\frac{2(1+\beta)}{p-1}+\frac{\alpha(1+\beta+\gamma)}{2-(\delta+\alpha)}}, & \text{if } \frac{\alpha(p+1)}{(p-1)} > n \\ C(1+t)^{-\frac{2(1+\beta)}{p-1}+\frac{\alpha(1+\beta+\gamma)}{2-(\delta+\alpha)}}\log(2+t), & \text{if } \frac{\alpha(p+1)}{(p-1)} = n \\ C(1+t)^{-\frac{2(1+\beta)}{p-1}+\frac{1+\beta+\gamma}{2-(\delta+\alpha)}(n-\frac{2\alpha}{p-1})}, & \text{if } \frac{\alpha(p+1)}{(p-1)} < n. \end{cases}$$

Remark 2. *The existence result can be proved using the same technique as in [8] where in this case the Caffarelli-Kohn-Nirenberg inequality is used instead of the Gagliardo-Nirenberg inequality, with the additional consideration of the inequality* $|x|^\delta \le (1+|x|^2)^{\frac{\delta}{2}}$. *Hence, we omit the proof here.*

Proof. [Proof of Theorem 3.1] We split the proof into three parts, the first part considers the case $x \in \Omega(t, L, t_0)$, the second part covers the case $x \in \Omega^c(t, L, t_0)$ and the third part combines the two results . We state the result in each of the zones in the form of a lemma.

Case 1: $(x \in \Omega(t, L, t_0))$. In this region, we define a function $E_\psi(\Omega(t, L, t_0))$ by

$$(3.3) \qquad E_\psi(\Omega(t, L, t_0)) := (t_0+t)^{\beta+\frac{\alpha A}{2}}\mathcal{E}(t) + \nu\mathcal{H}(t)$$

where ν is a small positive constant to be determined later, and the functions $H_E(t;\Omega(t;L,t_0))$, $H_1(t)$ and $H_2(t)$ by

$$(3.4) H_E(t;\Omega(t;L,t_0)) := \int_{\Omega(t;L,t_0)} E_\psi(\Omega(t,L,t_0))dx$$

$$H_1(t) := \int_0^{2\pi} E_\psi(\quad \Omega(t,L,t_0))\Big|_{|x|=\sqrt{(t_0+t)^A-L}}\Big[(t_0+t)^A - L\Big]^{\frac{N-1}{2}} d\theta \\ \times \frac{d}{dt}\sqrt{(t_0+t)^A - L}$$

(3.5)

$$(3.6) \qquad H_2(t) := \int_{\partial\Omega(t;L,t_0)} e^{2\psi}\Big[(t_0+t)^{\beta+\frac{\alpha A}{2}}\rho(x)\nabla u u_t + \nu\rho(x)u\nabla u\Big]\cdot \overrightarrow{n} dS$$

where $\overrightarrow{n}$ is the unit outward normal vector of $\partial\Omega(t;L,t_0)$. Then we state the next lemma.

Lemma 3.2. *Let u be a solution of (1.1) and the functions $\mathcal{E}(t)$ and $\mathcal{H}(t)$ be defined as in (2.6) and (2.7) above, then for $x \in \Omega(t, L, t_0)$, the function $E_\psi(\Omega(t, L, t_0))$ satisfies*

$$\begin{aligned}
&\tfrac{d}{dt}E_\psi(\Omega(t,L,t_0))\\
&\leq \nabla\cdot(e^{2\psi}\Big[(t_0+t)^{\beta+\frac{\alpha A}{2}}\rho(x)\nabla u u_t + \nu\rho(x)u\nabla u\Big])\\
&-k_0e^{2\psi}\eta(t)\Big[1+(t_0+t)^{\beta+\frac{\alpha A}{2}}(-\psi_t)\Big]\Big(|u_t|^2+a(t,x)|\nabla u|^2+|u|^{p+1}\Big)\\
&-k_0\Big[\tfrac{1}{(t_0+t)}+(-\psi_t)\Big]e^{2\psi}\eta(t)b(t,x)|u|^2 - k_0e^{2\psi}\eta(t)|u|^{p+1}
\end{aligned} \tag{3.7}$$

where k_0 is a positive constant to be determined later. Furthermore, we have

$$\begin{aligned}
&\tfrac{d}{dt}\quad\Big((t_0+t)^m H_E(t;\Omega(t;L,t_0))\Big] - (t_0+t)^m\Big(H_1(t)+H_2(t)\Big)\\
&\leq\begin{cases}
C(1+t)^{m-\gamma-\frac{(1+\beta)(p+1)}{p-1}}, & \text{if } \frac{\alpha(p+1)}{(p-1)}>n\\
C(1+t)^{m-\gamma-\frac{(1+\beta)(p+1)}{p-1}}\log(2+t), & \text{if } \frac{\alpha(p+1)}{(p-1)}=n\\
C(1+t)^{m-\gamma-\frac{(1+\beta)(p+1)}{p-1}+\frac{1+\beta+\gamma}{2-(\delta+\alpha)}(n-\frac{\alpha(p+1)}{p-1})}, & \text{if } \frac{\alpha(p+1)}{(p-1)}<n.
\end{cases}
\end{aligned} \tag{3.8}$$

Proof. Multiplying (2.8) by $(t_0+t)^{\beta+\frac{\alpha A}{2}}$, we obtain

$$\begin{aligned}
&\tfrac{d}{dt}\Big[(t_0+t)^{\beta+\frac{\alpha A}{2}}\mathcal{E}(t)\Big]\\
&\leq \nabla\cdot(e^{2\psi}(t_0+t)^{\beta+\frac{\alpha A}{2}}\rho(x)\nabla u u_t) + \tfrac{\eta_t(t)}{2}(t_0+t)^{\beta+\frac{\alpha A}{2}}|u_t|^2\\
&+\Big[\tfrac{(\beta+\frac{\alpha A}{2})}{2(t_0+t)^{1-(\beta+\frac{\alpha A}{2})}} - \tfrac{b(t,x)}{4}(t_0+t)^{\beta+\frac{\alpha A}{2}} + (t_0+t)^{\beta+\frac{\alpha A}{2}}\psi_t\Big]e^{2\psi}\eta(t)|u_t|^2\\
&+\Big[\tfrac{(\beta+\frac{\alpha A}{2})}{2(t_0+t)^{1-(\beta+\frac{\alpha A}{2})}} + \tfrac{\psi_t}{3}(t_0+t)^{\beta+\frac{\alpha A}{2}}\Big]e^{2\psi}\rho(x)|\nabla u|^2\\
&+\Big[\tfrac{(\beta+\frac{\alpha A}{2})-\gamma}{(p+1)(t_0+t)^{1-(\beta+\frac{\alpha A}{2})}} + \tfrac{2\psi_t}{p+1}(t_0+t)^{\beta+\frac{\alpha A}{2}}\Big]e^{2\psi}\eta(t)|u|^{p+1}.
\end{aligned} \tag{3.9}$$

Observe that $\beta + \frac{\alpha A}{2} \leq \beta + \alpha < 1$ since $A < 2$ and $\alpha + \beta + \delta + \gamma < 1$.

Now, multiplying (2.14) by ν (where $\nu < b_0$) and adding the resulting estimate to (3.9), we get

$$
\begin{aligned}
&\tfrac{d}{dt}\Big[(t_0+t)^{\beta+\frac{\alpha A}{2}}\mathcal{E}(t)+\nu\mathcal{H}(t)\Big]\\
&\le \nabla\cdot\big(e^{2\psi}\Big[(t_0+t)^{\beta+\frac{\alpha A}{2}}\rho(x)\nabla u u_t+\nu\rho(x)u\nabla u\Big]\big)\\
&+\Big[\tfrac{(\beta+\frac{\alpha A}{2})-\gamma(1-\frac{\nu}{b_0})}{2(t_0+t)^{1-(\beta+\frac{\alpha A}{2})}}+\nu-\tfrac{b_0}{4}+\tfrac{(\epsilon_1 b_0-3\nu)}{\epsilon_1 b_0}(t_0+t)^{\beta+\frac{\alpha A}{2}}\psi_t\Big]e^{2\psi}\eta(t)|u_t|^2\\
&+\Big[\tfrac{(\beta+\frac{\alpha A}{2})}{2(t_0+t)^{1-(\beta+\frac{\alpha A}{2})}}-\tfrac{\nu}{4}+\tfrac{\psi_t}{3}(t_0+t)^{\beta+\frac{\alpha A}{2}}\Big]e^{2\psi}\rho(x)|\nabla u|^2\\
&+\nu\Big[\tfrac{-\beta}{2(t_0+t)}+\tfrac{(1-\epsilon_1)}{3}\psi_t\Big]e^{2\psi}\eta(t)b(t,x)|u|^2\\
&+\Big[\tfrac{(\beta+\frac{\alpha A}{2})-\gamma}{(p+1)(t_0+t)^{1-(\beta+\frac{\alpha A}{2})}}-\nu+\tfrac{2\psi_t}{p+1}(t_0+t)^{\beta+\frac{\alpha A}{2}}\Big]e^{2\psi}\eta(t)|u|^{p+1},
\end{aligned}
\tag{3.10}
$$

where we have used Schwartz inequality to obtain the following estimates for the third and last term on the right hand side of (2.14) respectively:

$$
\begin{aligned}
|2\psi_t u_t u| &\le \tfrac{\epsilon_1 b(t,x)(-\psi_t)}{3}|u|^2+\tfrac{3(-\psi_t)}{\epsilon_1 b_0}(1+t)^{\beta}(1+|x|^2)^{\frac{\alpha}{2}}|u_t|^2\\
&\le \tfrac{-\epsilon_1 b(t,x)\psi_t}{3}|u|^2-\tfrac{3\psi_t}{\epsilon_1 b_0}(t_0+t)^{\beta+\frac{\alpha A}{2}}|u_t|^2
\end{aligned}
\tag{3.11}
$$

and

$$
\begin{aligned}
|\eta_t(t)u_t u| &\le \tfrac{-b(t,x)\eta_t(t)}{2}|u|^2-\tfrac{\eta_t(t)}{2b_0}(1+t)^{\beta}(1+|x|^2)^{\frac{\alpha}{2}}|u_t|^2\\
&\le \tfrac{-b(t,x)\eta_t(t)}{2}|u|^2-\tfrac{\eta_t(t)}{2b_0}(t_0+t)^{\beta+\frac{\alpha A}{2}}|u_t|^2.
\end{aligned}
\tag{3.12}
$$

By a suitable choice of ν sufficiently small as mentioned earlier, we can now choose a positive constant k_0 such that the estimates below are satisfied

$$
\begin{aligned}
&\tfrac{(\beta+\frac{\alpha A}{2})-\gamma(1-\frac{\nu}{b_0})}{2t_0^{1-(\beta+\frac{\alpha A}{2})}}+\nu-\tfrac{b_0}{4}\le -k_0\\
&\tfrac{(\beta+\frac{\alpha A}{2})}{2t_0^{1-(\beta+\frac{\alpha A}{2})}}-\tfrac{\nu}{4}\le -k_0,\qquad \tfrac{(\beta+\frac{\alpha A}{2})-\gamma}{(p+1)t_0^{1-(\beta+\frac{\alpha A}{2})}}-\nu\le -2k_0\\
&\nu\tfrac{1-\epsilon_1}{3}\ge k_0,\quad \tfrac{(\epsilon_1 b_0-3\nu)}{\epsilon_1 b_0}\ge k_0,\quad \tfrac{1}{3}\ge k_0,\quad \tfrac{2}{(p+1)}\ge k_0,\quad \nu\tfrac{\beta}{2}\ge k_0,
\end{aligned}
\tag{3.13}
$$

this gives the desired estimate (3.7).

We now integrate the estimate (3.7) over $\Omega(t;L,t_0)$ to obtain

$$
\frac{d}{dt}H_E(t;\Omega(t;L,t_0))-H_1(t)-H_2(t)\le -H_3(t;\Omega(t;L,t_0)),
\tag{3.14}
$$

where

$$\begin{aligned} H_3 &\quad (t;\Omega(t;L,t_0)) \\ := &\quad k_0 \int_{\Omega(t;L,t_0)} e^{2\psi}\eta(t)\Big[(1+(-\psi_t)(t_0+t)^{\beta+\frac{\alpha A}{2}})|u_t|^2 + (1+(-\psi_t)(t_0+t)^{\beta+\frac{\alpha A}{2}}) \\ &\quad a(t,x)|\nabla u|^2 \\ &\quad +\Big(-\psi_t+\tfrac{1}{t_0+t}\Big)b(t,x)|u|^2 + (1+(-\psi_t)(t_0+t)^{\beta+\frac{\alpha A}{2}})|u|^{p+1} + |u|^{p+1}\Big]dx. \end{aligned} \tag{3.15}$$

Define the function $\mathcal{H}_{\mathcal{E}}$ by

$$\begin{aligned} &\mathcal{H}_{\mathcal{E}}(t;\Omega(t;L,t_0)) := \int_{\Omega(t;L,t_0)} \eta(t) \\ &\Big[(t_0+t)^{\beta+\frac{\alpha A}{2}}\Big[|u_t|^2 + a(t,x)|\nabla u|^2 + |u|^{p+1}\Big] + b(t,x)|u|^2\Big]e^{2\psi}dx. \end{aligned} \tag{3.16}$$

It can be proved easily that for positive constants k_1, k_2, the following inequality is satisfied:

$$k_1\mathcal{H}_{\mathcal{E}} \le H_E(t;\Omega(t;L,t_0)) \le k_2\mathcal{H}_{\mathcal{E}}. \tag{3.17}$$

Now, multiplying (3.14) by $(t_0+t)^m$ for m a constant which will be determined later, we obtain

$$\begin{aligned} \frac{d}{dt} &\quad \Big((t_0+t)^m H_E(t;\Omega(t;L,t_0))\Big] - (t_0+t)^m\Big(H_1(t)+H_2(t)\Big) \\ &\quad \le (t_0+t)^m\Big[\tfrac{m}{t_0+t}H_E(t;\Omega(t;L,t_0)) - H_3(t;\Omega(t;L,t_0))\Big]. \end{aligned} \tag{3.18}$$

The term on the right hand side is estimated as

$$\begin{aligned} \frac{m}{t_0+t} &\quad H_E(t;\Omega(t;L,t_0)) - H_3(t;\Omega(t;L,t_0)) \\ &\le \tfrac{mk_2}{t_0+t}\mathcal{H}_{\mathcal{E}}(t;\Omega(t;L,t_0)) - H_3(t;\Omega(t;L,t_0)) \\ &\le \int_{\Omega(t;L,t_0)} e^{2\psi}\eta(t)\Big[\frac{mk_2}{(t_0+t)^{1-(\beta+\frac{\alpha A}{2})}} - k_0\Big]\Big[|u_t|^2 + a(t,x)|\nabla u|^2 + |u|^{p+1}\Big]dx \\ &\quad + \int_{\Omega(t;L,t_0)} e^{2\psi}\eta(t)\Big[\Big[\tfrac{mk_2}{t_0+t}\Big]b(t,x)u^2 - k_0|u|^{p+1}\Big]dx, \end{aligned} \tag{3.19}$$

where we have used $\psi_t \leq 0$.
From (3.13), it can be easily seen that we can choose t_0 large enough, such that $\frac{mk_2}{t_0^{1-(\beta+\frac{\alpha A}{2})}} < \frac{k_0}{2}$. Therefore, the first term on the right hand side of (3.19) yields

$$\int\limits_{\Omega(t;L,t_0)} e^{2\psi}\eta(t)\left[\frac{mk_2}{(t_0+t)^{1-(\beta+\frac{\alpha A}{2})}} - k_0\right]\left[|u_t|^2 + a(t,x)|\nabla u|^2 + |u|^{p+1}\right]dx$$
$$\leq -\frac{k_0}{2}\int\limits_{\Omega(t;L,t_0)} e^{2\psi}\eta(t)(|u_t|^2 + a(t,x)|\nabla u|^2 + |u|^{p+1})dx \leq 0. \tag{3.20}$$

To estimate the second term on the right hand of (3.19), we apply Young's inequality to obtain

$$\int\limits_{\Omega(t;L,t_0)} e^{2\psi}\eta(t)\left[\left[\frac{mk_2}{t_0+t}\right]b(t,x)u^2 - k_0|u|^{p+1}\right]dx$$
$$\leq \int\limits_{\Omega(t;L,t_0)} e^{2\psi}\eta(t)\left[\left[\frac{mk_2}{(1+t)^{1+\beta}}\right]b_0(1+|x|^2)^{\frac{-\alpha}{2}}|u|^2 - k_0|u|^{p+1}\right]dx$$
$$\leq \int\limits_{\Omega(t;L,t_0)} e^{2\psi}\eta(t)\left[C(1+t)^{\frac{-(1+\beta)(p+1)}{p-1}}(1+|x|^2)^{\frac{-\alpha(p+1)}{2(p-1)}} - k_p|u|^{p+1}\right]dx$$
$$\leq C\eta(t)(1+t)^{-\frac{(1+\beta)(p+1)}{p-1}}\int\limits_{\Omega(t;L,t_0)} e^{2\psi}(1+|x|^2)^{\frac{-\alpha(p+1)}{2(p-1)}}dx$$
$$\leq C\eta(t)(1+t)^{-\frac{(1+\beta)(p+1)}{p-1}}\int_0^{(t_0+t)^{\frac{A}{2}}}\left(1+r^2\right)^{\frac{-\alpha(p+1)}{2(p-1)}}r^{n-1}dr \tag{3.21}$$

where $C = C(m, b_0, k_2, p)$ and $k_p = k_p(k_0, p)$. Define J by

$$J := C(1+t)^{-\frac{(1+\beta)(p+1)}{p-1}-\gamma}\int_0^{(t_0+t)^{\frac{A}{2}}}\left(1+r^2\right)^{\frac{-\alpha(p+1)}{2(p-1)}}r^{n-1}dr.$$

Thus, if $\frac{\alpha(p+1)}{(p-1)} > n$, it follows that

$$J \leq C(1+t)^{-\frac{(1+\beta)(p+1)}{p-1}-\gamma}, \tag{3.22}$$

if $\frac{\alpha(p+1)}{(p-1)} = n$, we have

$$J \leq C(1+t)^{-\frac{(1+\beta)(p+1)}{p-1}-\gamma}\log(2+t) \tag{3.23}$$

and if $\frac{\alpha(p+1)}{(p-1)} < n$, we obtain

$$J \leq C(1+t)^{-\frac{(1+\beta)(p+1)}{p-1}-\gamma+\frac{1+\beta+\gamma}{2-(\delta+\alpha)}(n-\frac{\alpha(p+1)}{p-1})}. \tag{3.24}$$

Combining (3.19) - (3.24), we have

$$\begin{aligned}&\frac{m}{t_0+t}H_E(t;\Omega(t;L,t_0)) - H_3(t;\Omega(t;L,t_0))\\ &\leq \begin{cases} C(1+t)^{-\frac{(1+\beta)(p+1)}{p-1}-\gamma}, & \text{if } \frac{\alpha(p+1)}{(p-1)} > n\\ C(1+t)^{-\frac{(1+\beta)(p+1)}{p-1}-\gamma}\log(2+t), & \text{if } \frac{\alpha(p+1)}{(p-1)} = n\\ C(1+t)^{-\frac{(1+\beta)(p+1)}{p-1}-\gamma+\frac{1+\beta+\gamma}{2-(\delta+\alpha)}(n-\frac{\alpha(p+1)}{p-1})}, & \text{if } \frac{\alpha(p+1)}{(p-1)} < n. \end{cases}\end{aligned} \tag{3.25}$$

Hence, we have that

$$\begin{aligned}&\frac{d}{dt}\Big((t_0+t)^m H_E(t;\Omega(t;L,t_0))\Big] - (t_0+t)^m\Big(H_1(t)+H_2(t)\Big)\\ &\leq \begin{cases} C(1+t)^{m-\gamma-\frac{(1+\beta)(p+1)}{p-1}}, & \text{if } \frac{\alpha(p+1)}{(p-1)} > n\\ C(1+t)^{m-\gamma-\frac{(1+\beta)(p+1)}{p-1}}\log(2+t), & \text{if } \frac{\alpha(p+1)}{(p-1)} = n\\ C(1+t)^{m-\gamma-\frac{(1+\beta)(p+1)}{p-1}+\frac{1+\beta+\gamma}{2-(\delta+\alpha)}(n-\frac{\alpha(p+1)}{p-1})}, & \text{if } \frac{\alpha(p+1)}{(p-1)} < n. \end{cases}\end{aligned} \tag{3.26}$$

□

Case 2: For the region $\Omega^c(t;L,t_0) = \Big\{x|(t_0+t)^A \leq L+|x|^2\Big\}$, we define another function $E_\psi(\Omega^c(t,L,t_0))$ by

$$E_\psi(\Omega^c(t,L,t_0)) := (L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\mathcal{E}(t) + \nu\mathcal{H}(t), \tag{3.27}$$

where ν is a small positive constant to be determined later. In addition, define

$$H_E(t;\Omega^c(t;L,t_0)) := \int_{\Omega^c(t;L,t_0)} E_\psi(\Omega^c(t,L,t_0))dx \tag{3.28}$$

$$\begin{aligned}H_1^*(t) := \int_0^{2\pi} E_\psi(&\Omega^c(t,L,t_0))\Big|_{|x|=\sqrt{(t_0+t)^A-L}}\Big[(t_0+t)^A - L\Big]^{\frac{N-1}{2}}d\theta\\ &\times\frac{d}{dt}\sqrt{(t_0+t)^A-L}\end{aligned} \tag{3.29}$$

$$H_2^*(t) := \int_{\partial\Omega^c(t;L,t_0)} e^{2\psi}\Big[(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\rho(x)\nabla u u_t + \nu\rho(x)u\nabla u\Big]\cdot \overrightarrow{n}\, dS$$

(3.30)

where $\overrightarrow{n}$ is the unit outward normal vector of $\partial\Omega^c(t;L,t_0)$.

We can now state the next lemma.

Lemma 3.3. *Let u be a solution of (1.1) and the functions $\mathcal{E}(t)$ and $\mathcal{H}(t)$ be defined as in (2.6) and (2.7) above, then for $x \in \Omega^c(t;L,t_0)$, the function $E_\psi(\Omega^c(t,L,t_0))$ satisfies*

$$\begin{aligned}
&\tfrac{d}{dt}E_\psi(\Omega^c(t,L,t_0))\\
&\leq \nabla\cdot(e^{2\psi}\Big[(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\rho(x)\nabla u u_t + \nu\rho(x)u\nabla u\Big])\\
&-k_0 e^{2\psi}\eta(t)\Big[1+(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}(-\psi_t)\Big]\Big(|u_t|^2 + a(t,x)|\nabla u|^2 + |u|^{p+1}\Big)\\
&-k_0\Big[\tfrac{1}{(t_0+t)} + (-\psi_t)\Big]e^{2\psi}\eta(t)b(t,x)|u|^2 - k_0[1+(L+|x|^2)^{-\frac{1}{A}[1-(\beta+\frac{\alpha A}{2})]}]\\
&e^{2\psi}\eta(t)|u|^{p+1}
\end{aligned}$$

(3.31)

where k_0 is a positive constant to be determined later. Moreover, we have that

$$\frac{d}{dt}\Big[(t_0+t)^m H_E(t;\Omega^c(t;L,t_0))\Big] - (t_0+t)^m\Big(H_1(t)+H_2(t)\Big) \leq 0.$$

(3.32)

Proof. Multiplying (2.8) by $(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}$, we obtain

$$\begin{aligned}
&\tfrac{d}{dt}\Big[(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\mathcal{E}(t)\Big]\\
&\leq \nabla\cdot(e^{2\psi}(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\rho(x)\nabla u u_t) + e^{2\psi}\tfrac{\eta_t(t)}{2}(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}|u_t|^2\\
&+\eta(t)\Big[-\tfrac{b(t,x)}{4}(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})} + (L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\psi_t\Big]e^{2\psi}|u_t|^2\\
&+\Big[(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\tfrac{\psi_t}{3}\Big]e^{2\psi}\rho(x)|\nabla u|^2 - \frac{\frac{1}{A}(\beta+\frac{\alpha A}{2})}{(L+|x|^2)^{1-\frac{1}{A}(\beta+\frac{\alpha A}{2})}}e^{2\psi}x\cdot\rho(x)\nabla u u_t\\
&+e^{2\psi}\eta(t)\Big[\tfrac{-\gamma(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}}{(p+1)(1+t)} + \tfrac{2\psi_t}{p+1}(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\Big]|u|^{p+1}.
\end{aligned}$$

(3.33)

Adding (3.33) to $\nu\times$ (2.19), we obtain

$$\begin{aligned}
&\tfrac{d}{dt}E_\psi(\Omega^c(t,L,t_0))\\
&\le \nabla\cdot\Big(e^{2\psi}\Big[(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\rho(x)\nabla u u_t+\nu\rho(x)u\nabla u\Big]\Big)\\
&-\tfrac{1}{A}(\beta+\tfrac{\alpha A}{2})e^{2\psi}(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})-1}x\cdot\rho(x)\nabla u u_t+\nu e^{2\psi}\tfrac{\eta_t(t)b(t,x)}{2}|u|^2\\
&+\eta(t)\Big[\nu-\tfrac{b(t,x)}{4}(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}+(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\psi_t\Big]e^{2\psi}|u_t|^2\\
&+\Big[-\tfrac{\nu}{4}+(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\tfrac{\psi_t}{3}\Big]e^{2\psi}\rho(x)|\nabla u|^2+e^{2\psi}\tfrac{\eta_t(t)}{2}(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}|u_t|^2\\
&+\eta(t)\Big[-\nu-\tfrac{\gamma(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}}{(p+1)(1+t)}+\tfrac{2\psi_t}{p+1}(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\Big]e^{2\psi}|u|^{p+1}\\
&+\nu\Big[\tfrac{-\beta}{2(t_0+t)}+\tfrac{\psi_t}{3}\Big]e^{2\psi}\eta(t)b(t,x)|u|^2+2\nu e^{2\psi}\eta(t)\psi_t u u_t+\nu e^{2\psi}\eta_t(t)u u_t.
\end{aligned}\tag{3.34}$$

For the second term on the right hand of (3.34), by using Schwartz inequality, we obtain

$$\begin{aligned}
|\tfrac{1}{A}(\beta &+\tfrac{\alpha A}{2})(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})-1}x\cdot\rho(x)\nabla u u_t|\\
&\le \tfrac{1}{A}(\beta+\tfrac{\alpha A}{2})(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})-\frac{1}{2}}|u_t|\rho(x)|\nabla u|\\
&\le \frac{\frac{1}{A}(\beta+\frac{\alpha A}{2})\rho(x)}{2(L+|x|^2)^{1-\frac{1}{A}(\beta+1+\frac{\alpha A}{2})}}\rho(x)|\nabla u|^2+\frac{\frac{1}{A}(\beta+\frac{\alpha A}{2})}{2(L+|x|^2)^{\frac{1}{A}[1-(\beta+\frac{\alpha A}{2})]}}|u_t|^2\\
&\le \frac{\frac{1}{A}(\beta+\frac{\alpha A}{2})\rho_0}{2(L+|x|^2)^{1-\frac{1}{A}(\beta+1+\frac{(\alpha+\delta)A}{2})}}\rho(x)|\nabla u|^2+\frac{\frac{1}{A}(\beta+\frac{\alpha A}{2})}{2(L+|x|^2)^{\frac{1}{A}[1-(\beta+\frac{\alpha A}{2})]}}|u_t|^2
\end{aligned}\tag{3.35}$$

and observe here that $\frac{1}{A}(\beta+1+\frac{(\alpha+\delta)A}{2})=\frac{2(\beta+1)+\gamma(\alpha+\delta)}{2(1+\beta+\gamma)}<1$. Also, by using the Schwartz inequality, we obtain the following estimates for the second to the last term and the last term on the right hand side of (3.34) respectively:

$$\begin{aligned}
|2\psi_t u u_t| &\le \tfrac{\epsilon_2}{3}(-\psi_t)b(t,x)|u|^2+\tfrac{3}{\epsilon_2 b_0}(-\psi_t)(1+t)^\beta(1+|x|^2)^{\frac{\alpha}{2}}|u_t|^2\\
&\le \tfrac{-\epsilon_2}{3}(\psi_t)b(t,x)|u|^2-\tfrac{3}{\epsilon_2 b_0}(\psi_t)(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}|u_t|^2
\end{aligned}\tag{3.36}$$

and

$$\begin{aligned}
|\eta_t(t)u_t u| &\le \tfrac{b(t,x)(-\eta_t(t))}{2}|u|^2+\tfrac{(-\eta_t(t))}{2b_0}(1+t)^\beta(1+|x|^2)^{\frac{\alpha}{2}}|u_t|^2\\
&\le \tfrac{-b(t,x)\eta_t(t)}{2}|u|^2-\tfrac{\eta_t(t)}{2b_0}(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}|u_t|^2.
\end{aligned}\tag{3.37}$$

Therefore, substituting the estimates (3.35) - (3.37) in (3.34), we get

$$
\begin{aligned}
&\tfrac{d}{dt}E_\psi(\Omega^c(t,L,t_0))\\
&\leq \nabla\cdot\Big(e^{2\psi}\Big[(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\rho(x)\nabla u u_t+\nu\rho(x)u\nabla u\Big]\Big)\\
&+\eta(t)\Bigg[\nu+\frac{\frac{1}{A}(\beta+\frac{\alpha A}{2})-\gamma(1-\frac{\nu}{b_0})}{2L^{\frac{1}{A}[1-(\beta+\frac{\alpha A}{2})]}}-\frac{b_0}{4}+(1-\frac{3\nu}{\epsilon_2 b_0})(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\psi_t\Bigg]e^{2\psi}|u_t|^2\\
&+\Bigg[-\frac{\nu}{4}+\frac{\frac{1}{A}(\beta+\frac{\alpha A}{2})\rho_0}{2L^{1-\frac{1}{A}(\beta+1+\frac{(\alpha+\delta)A}{2})}}+(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\frac{\psi_t}{3}\Bigg]e^{2\psi}\rho(x)|\nabla u|^2\\
&+\eta(t)\Bigg[-\nu-\frac{\gamma}{(p+1)(L+|x|^2)^{\frac{1}{A}[1-(\beta+\frac{\alpha A}{2})]}}+\frac{2\psi_t}{p+1}(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\Bigg]e^{2\psi}|u|^{p+1}\\
&+\nu\Bigg[\frac{-\beta}{2(t_0+t)}+\frac{(1-\epsilon_2)}{3}\psi_t\Bigg]e^{2\psi}\eta(t)b(t,x)|u|^2.
\end{aligned}
\tag{3.38}
$$

Now, just as in the Case 1, we choose a suitable value for ν which is sufficiently small and a positive constant k_0 such that the estimates we have below are satisfied.

$$
\begin{aligned}
&\nu+\frac{\frac{1}{A}(\beta+\frac{\alpha A}{2})-\gamma(1-\frac{\nu}{b_0})}{2L^{\frac{1}{A}[1-(\beta+\frac{\alpha A}{2})]}}-\frac{b_0}{4}\leq -k_0,\quad -\frac{\nu}{4}+\frac{\frac{1}{A}(\beta+\frac{\alpha A}{2})\rho_0}{2L^{1-\frac{1}{A}(\beta+1+\frac{(\alpha+\delta)A}{2})}}\leq -k_0,\\
&\nu\frac{(1-\epsilon_2)}{3}\geq k_0,\quad \frac{2}{p+1}\geq k_0,\quad \frac{1}{3}\geq k_0,\quad (1-\frac{3\nu}{\epsilon_2 b_0})\geq k_0,\quad \nu\geq 2k_0,\\
&\frac{\beta v}{2}\geq k_0,\quad \frac{\gamma}{p+1}\geq k_0,
\end{aligned}
\tag{3.39}
$$

which gives the desired estimate. Therefore by integrating the estimate (3.31) over $\Omega^c(t,L,t_0)$, we obtain

$$
\frac{d}{dt}H_E(t;\Omega^c(t;L,t_0))-H_1^*(t)-H_2^*(t)\leq -H_3(t;\Omega^c(t;L,t_0))
\tag{3.40}
$$

where

$$
\begin{aligned}
&H_3(t;\Omega^c(t;L,t_0))\\
&:=k_0\int_{\Omega^c(t;L,t_0)}\eta(t)e^{2\psi}\Big[\Big[1+(-\psi_t)(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\Big]\\
&\Big[|u_t|^2+a(t,x)|\nabla u|^2+|u|^{p+1}\Big]\\
&+\Big(-\psi_t+\frac{1}{t_0+t}\Big)b(t,x)|u|^2+[1+(L+|x|^2)^{-\frac{1}{A}[1-(\beta+\frac{\alpha A}{2})]}]|u|^{p+1}\Big]dx
\end{aligned}
\tag{3.41}
$$

Define the function $\mathcal{H}_{\mathcal{E}^c}$ by

$$\mathcal{H}_{\mathcal{E}^c} = \int\limits_{\Omega^c(t;L,t_0)} \eta(t)\Big[(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\Big[|u_t|^2 + a(t,x)|\nabla u|^2 + |u|^{p+1}\Big] + b(t,x)|u|^2\Big]e^{2\psi}dx. \tag{3.42}$$

It can be proved in a similar way as in Case 1 that for positive constants k_1^*, k_2^*, the following inequality holds.

$$k_1^*\mathcal{H}_{\mathcal{E}^c} \leq H_E(t;\Omega^c(t;L,t_0)) \leq k_2^*\mathcal{H}_{\mathcal{E}^c}. \tag{3.43}$$

Multiplying (3.40) by $(t_0+t)^m$ for the same constant m as in Case 1, we have

$$\begin{aligned} \frac{d}{dt} & \Big[(t_0+t)^m H_E(t;\Omega^c(t;L,t_0))\Big] - (t_0+t)^m\Big(H_1^*(t)+H_2^*(t)\Big) \\ & \leq (t_0+t)^m\Big[\frac{m}{t_0+t}H_E(t;\Omega^c(t;L,t_0)) - H_3(t;\Omega^c(t;L,t_0))\Big]. \end{aligned} \tag{3.44}$$

The term on the right hand side is estimated as

$$\begin{aligned} \frac{m}{t_0+t} & H_E(t;\Omega^c(t;L,t_0)) - H_3(t;\Omega^c(t;L,t_0)) \\ & \leq \frac{mk_2^*}{t_0+t}\mathcal{H}_{\mathcal{E}^c} - H_3(t;\Omega^c(t;L,t_0)) \\ & \leq \int\limits_{\Omega^c(t;L,t_0)} e^{2\psi}\Big[\frac{mk_2^*(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}}{(t_0+t)} - k_0\Big[1+(-\psi_t)(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\Big]\Big] \\ & \qquad \times\eta(t)\Big[|u_t|^2 + a(t,x)|\nabla u|^2 + |u|^{p+1}\Big]dx \\ & + \int\limits_{\Omega^c(t;L,t_0)} e^{2\psi}\eta(t)\Big[\Big(\frac{mk_2^*}{t_0+t} - k_0(-\psi_t)\Big)b(t,x)u^2 - k_0|u|^{p+1}\Big]dx. \end{aligned} \tag{3.45}$$

It can be seen from (3.39) that we can suitably choose k_0 such that $mk_2^* \leq \lambda k_0(1+\beta+\gamma)$. Therefore the first term on the right hand side of (3.45) yields

$$\int_{\Omega^c(t;L,t_0)} e^{2\psi}(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}\Big[\frac{mk_2^*}{(t_0+t)} - k_0\lambda(1+\beta+\gamma)\frac{(L+|x|^2)^{\frac{2-(\delta+\alpha)}{2}}}{(t_0+t)^{2+\beta+\gamma}}\Big]$$
$$\times\eta(t)\Big[|u_t|^2 + a(t,x)|\nabla u|^2 + |u|^{p+1}\Big]dx$$
$$\leq \int_{\Omega^c(t;L,t_0)} e^{2\psi}\frac{(L+|x|^2)^{\frac{1}{A}(\beta+\frac{\alpha A}{2})}}{(t_0+t)}\Big[mk_2^* - k_0\lambda(1+\beta+\gamma)\Big]$$
$$\times\eta(t)\Big[|u_t|^2 + a(t,x)|\nabla u|^2 + |u|^{p+1}\Big]dx \leq 0. \tag{3.46}$$

Likewise, for the second term on the right hand side of (3.45), we have

$$\int_{\Omega^c(t;L,t_0)} e^{2\psi}\eta(t)\Big[\Big(\frac{mk_2^*}{t_0+t} - k_0\lambda(1+\beta+\gamma)\frac{(L+|x|^2)^{\frac{2-(\alpha+\delta)}{2}}}{(t_0+t)^{2+\beta+\gamma}}\Big)b(t,x)u^2 - k_0|u|^{p+1}\Big]dx$$
$$\leq \int_{\Omega^c(t;L,t_0)} e^{2\psi}\eta(t)\Big[\Big(\frac{mk_2^*}{t_0+t} - \frac{k_0\lambda(1+\beta+\gamma)}{(t_0+t)}\Big)b(t,x)u^2\Big]dx \leq 0. \tag{3.47}$$

Consequently, we have

$$\frac{d}{dt}\Big[(t_0+t)^m H_E(t;\Omega^c(t;L,t_0))\Big] - (t_0+t)^m\Big(H_1^*(t) + H_2^*(t)\Big) \leq 0. \tag{3.48}$$

□

Case 3. With $t_0 > L$ and $H_1 = H_1^*$, $H_2 = H_2^*$, then it follows from (3.26) and (3.48) that

$$\frac{d}{dt}\Big((t_0+t)^m\Big[H_E(t;\Omega(t;L,t_0)) + H_E(t;\Omega^c(t;L,t_0))\Big]\Big)$$
$$\leq \begin{cases} C(1+t)^{m-\gamma-\frac{(1+\beta)(p+1)}{p-1}}, & \text{if } \frac{\alpha(p+1)}{(p-1)} > n \\ C(1+t)^{m-\gamma-\frac{(1+\beta)(p+1)}{p-1}}\log(2+t), & \text{if } \frac{\alpha(p+1)}{(p-1)} = n \\ C(1+t)^{m-\gamma-\frac{(1+\beta)(p+1)}{p-1}+\frac{1+\beta+\gamma}{2-(\delta+\alpha)}(n-\frac{\alpha(p+1)}{p-1})}, & \text{if } \frac{\alpha(p+1)}{(p-1)} < n. \end{cases} \tag{3.49}$$

Choosing

$$m = \begin{cases} \frac{(1+\beta)(p+1)}{p-1} - 1 + \gamma + \epsilon & \text{if } \frac{\alpha(p+1)}{(p-1)} > n \\ \frac{(1+\beta)(p+1)}{p-1} - \frac{1+\beta+\gamma}{2-(\delta+\alpha)}(n - \frac{\alpha(p+1)}{p-1}) - 1 + \gamma + \epsilon & \text{if } \frac{\alpha(p+1)}{(p-1)} < n, \end{cases} \tag{3.50}$$

for $0 < \epsilon < 1$ and integrating (3.49) over $[0, t]$, we obtain

$$\Big[H_E(t;\ \Omega(t;L,t_0)) + H_E(t;\Omega^c(t;L,t_0))\Big] \leq \begin{cases} C(1+t)^{-\frac{(1+\beta)(p+1)}{p-1}+1-\gamma}, & \text{if } \frac{\alpha(p+1)}{(p-1)} > n \\ C(1+t)^{-\frac{(1+\beta)(p+1)}{p-1}+1-\gamma}\log(2+t), & \text{if } \frac{\alpha(p+1)}{(p-1)} = n \\ C(1+t)^{-\frac{(1+\beta)(p+1)}{p-1}+\frac{1+\beta+\gamma}{2-(\delta+\alpha)}(n-\frac{\alpha(p+1)}{p-1})+1-\gamma}, & \text{if } \frac{\alpha(p+1)}{(p-1)} < n. \end{cases} \tag{3.51}$$

In particular, we have

$$\begin{aligned} \mathcal{A} := & \int_{\Omega(t;L,t_0)} e^{2\psi} b(t,x)|u|^2 dx + \int_{\Omega^c(t;L,t_0)} e^{2\psi} b(t,x)|u|^2 dx \\ \leq & \begin{cases} C(1+t)^{-\frac{(1+\beta)(p+1)}{p-1}+1}, & \text{if } \frac{\alpha(p+1)}{(p-1)} > n \\ C(1+t)^{-\frac{(1+\beta)(p+1)}{p-1}+1}\log(2+t), & \text{if } \frac{\alpha(p+1)}{(p-1)} = n \\ C(1+t)^{-\frac{(1+\beta)(p+1)}{p-1}+\frac{1+\beta+\gamma}{2-(\delta+\alpha)}(n-\frac{\alpha(p+1)}{p-1})+1}, & \text{if } \frac{\alpha(p+1)}{(p-1)} < n. \end{cases} \end{aligned} \tag{3.52}$$

Now, set $y = \frac{(L+|x|^2)^{\frac{2-(\delta+\alpha)}{2}}}{(t_0+t)^{1+\beta+\gamma}}$. Since the following estimate

$$(1+|x|^2)^{\frac{-\alpha}{2}} \geq (L+|x|^2)^{\frac{-\alpha}{2}} = \left[\frac{(L+|x|^2)^{\frac{2-(\delta+\alpha)}{2}}}{(t_0+t)^{1+\beta+\gamma}}\right]^{\frac{-\alpha}{2-(\delta+\alpha)}} (t_0+t)^{\frac{-\alpha}{2-(\delta+\alpha)}(1+\beta+\gamma)} \tag{3.53}$$

holds, then for $y > 0$, we have that

$$e^{2\lambda y} y^{-\frac{\alpha}{2-(\delta+\alpha)}} \geq C. \tag{3.54}$$

Therefore, we obtain

$$\mathcal{A} \geq C(1+t)^{-\beta-\frac{\alpha}{2-(\delta+\alpha)}(1+\beta+\gamma)} \int_{\mathbf{R}^N} u^2 dx \tag{3.55}$$

which gives the desired estimate. □

Remark 3. *The decay result in Theorem 3.1 coincides with that of [8] for the case* $\delta = \gamma = 0$ *and with that of [13] for the case* $\delta = \gamma = \alpha = 0$.

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